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Composite variety-based topological theories

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Motivated by the recent result of S. E. Rodabaugh [2] on categorical redundancy of lattice-valued bitopology, we considered in [5] another viewpoint on the topic, based on the notion of composite variety-based topological theory. The new concept, apart from providing a variable-basis generalization of bitopology, incorporates the most important approaches to topology, currently developed in the fuzzy community, streamlining their categorically-algebraic properties void of point-set lattice theoretic dependencies. Having considered different ways of interaction between composite topology and topology, e.g., embedding the former into the latter as a full bicoreflective subcategory, we have finally arrived at the conclusion that (variable-basis) bitopological theories still deserved to be studied on their own. It is the purpose of the talk to discuss briefly the most important results obtained in the field so far.

The approach to fuzziness coined as variety-based topology [4] relies heavily on the notion of algebra. The structure is a set equipped with a family of operations defined on it, which satisfy certain identities, e.g., semigroup, monoid, group and also complete lattice, frame, quantale.

Definition 1 *Let* $\Omega = (n_\lambda)_{\lambda\in\Lambda}$ *be a (possibly proper) class of cardinal numbers. An* Ω-algebra *is a pair* $(A, (\omega_\lambda^A)_{\lambda\in\Lambda})$ *consisting of a set* A *and a family of maps* $A^{n_\lambda} \xrightarrow{\omega_\lambda^A} A$*, called* n_λ-ary operations *on* A*. An* Ω-homomorphism $(A, (\omega_\lambda^A)_{\lambda\in\Lambda}) \xrightarrow{f} (B, (\omega_\lambda^B)_{\lambda\in\Lambda})$ *is a map* $A \xrightarrow{f} B$ *such that* $f \circ \omega_\lambda^A = \omega_\lambda^B \circ f^{n_\lambda}$ *for every* $\lambda \in \Lambda$*.* $\mathbf{Alg}(\Omega)$ *is the construct of* Ω*-algebras and* Ω*-homomorphisms.* $\mathcal{M}$ *(resp.* $\mathcal{E}$*) being the class of* Ω*-homomorphisms with injective (resp. surjective) underlying maps, a* variety of Ω-algebras *is a full subcategory of* $\mathbf{Alg}(\Omega)$ *closed under the formation of products,* $\mathcal{M}$*-subobjects (subalgebras) and* $\mathcal{E}$*-quotients (homomorphic images). The objects (resp. morphisms) of a variety are called* algebras *(resp.* homomorphisms*).*

The categorical dual of a variety $\mathbf{A}$ is denoted by $\mathbf{LoA}$ (the "**Lo**" comes from "localic"), whose objects (resp. morphisms) are called *localic algebras* (resp. *homomorphisms*). Given a morphism f of $\mathbf{A}$, the respective morphism of $\mathbf{LoA}$ is denoted by f^{op} and vice versa.

Variety-based topological theories provide a mixture of powerset theories of [3] and topological theories of [1], using our own results of [6]. Recall that given a map $X \xrightarrow{f} Y$, there exist the standard image and preimage operators on the respective powersets $\mathcal{P}(X) \xrightarrow{f^{\rightarrow}} \mathcal{P}(Y)$ and $\mathcal{P}(Y) \xrightarrow{f^{\leftarrow}} \mathcal{P}(X)$, defined by $f^{\rightarrow}(S) = \{f(x) \,|\, x \in S\}$ and $f^{\leftarrow}(T) = \{x \,|\, f(x) \in T\}$.

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Definition 2 *A* variety-based topological theory *in a category* $\mathbf{X}$ *is a functor* $\mathbf{X} \xrightarrow{T} \mathbf{LoA}$. *For each such theory* T *denote by* $\mathbf{Top}(T)$ *the concrete category over* $\mathbf{X}$, *whose objects (called* variety-based topological spaces*) are pairs* (X, τ) *with* X *an* $\mathbf{X}$*-object and* τ *a subalgebra of* $T(X)$ *(called* variety-based topology *on* X*), and whose morphisms* $(X_1, \tau_1) \xrightarrow{f} (X_2, \tau_2)$ *are those* $\mathbf{X}$*-morphisms* $X_1 \xrightarrow{f} X_2$ *that satisfy* $((Tf)^{op})^{\rightarrow}(\tau_2) \subseteq \tau_1$ *(called* continuity*). The underlying functor to the ground category* $\mathbf{X}$ *is defined by* $|(X_1, \tau_1) \xrightarrow{f} (X_2, \tau_2)| = X_1 \xrightarrow{f} X_2$.

The framework can be easily extended to composite variety-based topological theories. Briefly speaking, a set (or, more generally, an object of the ground category) is allowed to have a set-indexed family of topological structures, each of them based on its own underlying algebra, coming from a (possibly) different variety. Categorical background arises from the fact that given a set-indexed family $(\mathbf{X} \xrightarrow{T_i} \mathbf{LoA}_i)_{i \in I}$ of theories, the well-known construction of product categories [1] provides the functor $\mathbf{X} \xrightarrow{T_I} \prod_{i \in I} \mathbf{LoA}_i$, with the property $\mathbf{X} \xrightarrow{T_I} \prod_{i \in I} \mathbf{LoA}_i \xrightarrow{P_j} \mathbf{LoA}_j = \mathbf{X} \xrightarrow{T_j} \mathbf{LoA}_j$ for every $j \in I$ (P_j is the projection functor).

Definition 3 *Given a set-indexed family* $(\mathbf{X} \xrightarrow{T_i} \mathbf{LoA}_i)_{i \in I}$ *of variety-based topological theories, the* composite variety-based topological theory *in the category* $\mathbf{X}$ *is the functor* $\mathbf{X} \xrightarrow{T_I} \prod_{i \in I} \mathbf{LoA}_i$. *For each such theory* T_I *denote by* $\mathbf{CTop}(T_I)$ *the concrete category over* $\mathbf{X}$, *whose objects (called* composite variety-based topological spaces*) are pairs* $(X, (\tau_i)_{i \in I})$ *with* X *an* $\mathbf{X}$*-object and* τ_i *a subalgebra of* $T_i(X)$ *for every* $i \in I$ *(*$(\tau_i)_{i \in I}$ *is called* composite variety-based topology *on* X*), and whose morphisms* $(X, (\tau_i)_{i \in I}) \xrightarrow{f} (Y, (\sigma_i)_{i \in I})$ *are those* $\mathbf{X}$*-morphisms* $X \xrightarrow{f} Y$ *that satisfy* $((T_i f)^{op})^{\rightarrow}(\sigma_i) \subseteq \tau_i$ *for every* $i \in I$ *(called* composite continuity*). The underlying functor to the ground category* $\mathbf{X}$ *is defined by* $|(X, (\tau_i)_{i \in I}) \xrightarrow{f} (Y, (\sigma_i)_{i \in I})| = X \xrightarrow{f} Y$.

The composite topological theory induced by a source $\mathcal{S} = (T_{i_0} \xrightarrow{\eta_i} T_i)_{i \in I}$, with a fixed element $i_0 \in I$, is denoted by $\overleftarrow{T_I}$. The actual homomorphisms, given by the components of the natural transformations in question, go in the opposite direction, which is underlined by the arrow notation. The following introduces a machinery for gluing together topological theories.

Lemma 4 *Given a topological theory* $\overleftarrow{T_I}$, *there exists a functor* $\mathbf{X} \xrightarrow{T} \mathbf{LoA}_{i_0}$, $T(X \xrightarrow{f} Y) = \times_{i \in I} (\eta^{op}_{i_X})^{\rightarrow}(T_i(X)) \xrightarrow{Tf} \times_{i \in I} (\eta^{op}_{i_Y})^{\rightarrow}(T_i(Y))$, Tf *having the property* $\pi^X_i \circ (Tf)^{op} = (T_{i_0} f)^{op} \circ \pi^Y_i$ *for every* $i \in I$ *(*π_i *is the projection homomorphism).*

Definition 5 *Given a topological theory* $\overleftarrow{T_I}$, *the functor* T *of Lemma 4 is denoted by* $\overleftarrow{T^{\times}_I}$ *and called the* gluing functor *w.r.t.* $\overleftarrow{T_I}$.

The next theorem establishes a way of going from composite topology to topology. The attentive reader will easily infer that we have nearly missed the possibility of reducing composite topology to topology, making the former one redundant in fuzzy mathematics.

Theorem 6 *There exists a concrete functor* $\mathbf{CTop}(\overleftarrow{T_I}) \xrightarrow{E^{\rightarrow}_{\times}} \mathbf{Top}(\overleftarrow{T^{\times}_I})$, $E^{\rightarrow}_{\times}((X, (\tau_i)_{i \in I}) \xrightarrow{f} (Y, (\sigma_i)_{i \in I})) = (X, \times_{i \in I} (\eta^{op}_{i_X})^{\rightarrow}(\tau_i)) \xrightarrow{f} (Y, \times_{i \in I} (\eta^{op}_{i_Y})^{\rightarrow}(\sigma_i))$. *If the respective source consists of Injective-transformations and the signature of* $\mathbf{A}_{i_0}$ *has at least one nullary operation, then* $E^{\rightarrow}_{\times}$ *is a full embedding, which in general is not an isomorphism.*

Theorem 6 provides a plenty of good properties of the new functor. The following one shows a significant merit left outside, i.e., the existence of a right adjoint.

Theorem 7 *There exists a concrete functor* $\mathbf{Top}(\overleftarrow{T_I^{\times}}) \xrightarrow{F_{\times}^{\leftarrow}} \mathbf{CTop}(\overleftarrow{T_I})$*,* $F_{\times}^{\rightarrow}((X,\tau) \xrightarrow{f} (Y,\sigma)) = (X, ((\eta_{i_X}^{op})^{\leftarrow} \circ (\pi_i^X)^{\rightarrow}(\tau))_{i \in I}) \xrightarrow{f} (Y, ((\eta_{i_Y}^{op})^{\leftarrow} \circ (\pi_i^Y)^{\rightarrow}(\sigma))_{i \in I})$*, which in general is neither full nor an embedding. If the respective source consists of isomorphisms, then* $E_{\times}^{\rightarrow}$ *is a left-adjoint to* $F_{\times}^{\leftarrow}$*.* $E_{\times}^{\rightarrow}$ *is a right-inverse to* $F_{\times}^{\leftarrow}$ *provided that the source consists of Injective-transformations and the signature of* $\mathbf{A}_{i_0}$ *has at least one nullary operation.*

Corollary 8 $\mathbf{CTop}(\overleftarrow{T_I})$ *is isomorphic to a full bicoreflective subcategory of* $\mathbf{Top}(\overleftarrow{T_I^{\times}})$ *provided that the respective source consists of isomorphisms and the signature of* $\mathbf{A}_{i_0}$ *has at least one nullary operation.*

Meta-mathematically restated, composite topology can be fully embedded into topology. The sticking point, however, is the lack of a left adjoint for the embedding.

Lemma 9 *In general,* $E_{\times}^{\rightarrow}$ *does not preserve terminal objects and therefore has no left adjoint. It follows that the functor is not an equivalence.*

The results obtained undermine slightly the claim on categorical redundancy of lattice-valued bitopology in mathematics. The reason is that the main candidate of [2] for reducing bitopology to topology, the functor $E_{\times}^{\rightarrow}$, does not look as promising as before, having no left adjoint and indulged into problems with limits. Moreover, even in case of fixed-basis, the author of [2] already foresees some problems resulting from passing to a simpler topological structure (one topology instead of two) but a more complex lattice-theoretic one (the product lattice). Our variety-based approach clearly shows that the case of variable-basis results in additional difficulties, possibly incomparable with the advantage gained. By our opinion, it is better to preserve both frameworks, providing at the same time a nice way of interaction between them.

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