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CATEGORICALLY-ALGEBRAIC TOPOLOGY

SERGEJS SOLOVJOVS

This talk aims at presenting a new way of approaching topological structures, induced by recent developments in the field of fuzzy topology, and deemed to incorporate in itself both the crisp and the fuzzy settings. Based on category theory and universal algebra, the new framework is called *categorically-algebraic* (*catalg*), to underline its motivating theories, on one hand, and to distinguish it from the currently dominating in the fuzzy community *point-set lattice-theoretic* (*poslat*) *topology* of S. E. Rodabaugh [4], on the other.

At the bottom of the new approach lies a generalization of several aspects of the standard topological framework. Taking the case of classical topological spaces as an example, there are three main cornerstones in their theory.

1. The starting *backward powerset operator*, which can be represented as a functor $\mathbf{Set} \xrightarrow{(-)^{\leftarrow}} \mathbf{CBAlg}^{op}$ from the category of sets to the dual of the category of complete Boolean algebras, assigning to every set X its powerset 2^X, and to every map $X \xrightarrow{f} Y$ its extension $2^Y \xrightarrow{f^{\leftarrow}} 2^X$, $f^{\leftarrow}(\alpha) = \alpha \circ f$.

2. The induced *topological theory*, which describes the underlying algebraic structure of topological spaces, and which essentially is the composition of the powerset operator and the forgetful functor $\mathbf{CBAlg} \xrightarrow{\|-\|} \mathbf{Frm}$ to the category of frames.

3. The resulting category $\mathbf{Top}$ of *topological spaces*, which has pairs (X, τ), with τ a subframe of $\|2^X\|$, as objects, and maps $(X, \tau) \xrightarrow{f} (Y, \sigma)$, with $(f^{\leftarrow})^{\rightarrow}(\sigma) \subseteq \tau$, as morphisms (forward powerset operator $(-)^{\rightarrow}$ can be easily avoided).

The proposed catalg framework chooses the starting backward powerset operator to be a functor $\mathbf{X} \xrightarrow{P} \mathbf{LoA}$ to the dual category of some variety of algebras $\mathbf{A}$ (in an extended form, including, e.g., the case of quantales). The induced topological theory $\mathbf{X} \xrightarrow{T} \mathbf{LoB}$ is the composition of P and some reduct (in the obvious algebraic sense, but considered as a forgetful functor) $\mathbf{A} \xrightarrow{\|-\|} \mathbf{B}$ of $\mathbf{A}$. The resulting category $\mathbf{Top}(T)$ of topological structures (T*-spaces*) has then pairs (X, τ), with τ (T*-topology*) a subalgebra of $T(X)$, as objects, and $\mathbf{X}$-morphisms $(X, \tau) \xrightarrow{f} (Y, \sigma)$, with $((Tf)^{op})^{\rightarrow}(\sigma) \subseteq \tau$ (T*-continuity*), as morphisms.

The case of the powerset theory P having the form of $\mathbf{Set} \times \mathbf{LoA} \xrightarrow{(-)^{\leftarrow}} \mathbf{LoA}$, $((X_1, A_1) \xrightarrow{(f,\varphi)} (X_2, A_2))^{\leftarrow} = A_1^{X_1} \xrightarrow{((f,\varphi)^{\leftarrow})^{op}} A_2^{X_2}$, $(f, \varphi)^{\leftarrow}(\alpha) = \varphi^{op} \circ \alpha \circ f$ provides a rich source of examples, including almost all approaches to topology of the fuzzy community (taking in each case an appropriate variety $\mathbf{A}$). Moreover, one easily incorporates the non-standard example of *closure spaces* of D. Aerts *et al.* [1]. The still missing settings are included extending the machinery in two ways.

1. Replacing a single powerset theory P by a set-indexed family $(\mathbf{X} \xrightarrow{P_i} \mathbf{LoA}_i)_{i \in I}$ of theories, with the respective set $(\| - \|, \mathbf{B}_i)_{i \in I}$ of reducts, provides a *composite*

topological theory $\mathbf{X} \xrightarrow{T_I} \prod_{i \in I} \mathbf{LoB}_i$, resulting in the category $\mathbf{Top}(T_I)$ of *composite* topological structures, with objects pairs $(X, (\tau_i)_{i \in I})$, where τ_i is a subalgebra of $T_i(X)$ for every $i \in I$. The framework incorporates *bitopological spaces* of J. C. Kelly [2] as well as their fuzzy analogue of S. E. Rodabaugh [5].

2. Introducing the category **L-FA** of fuzzy **A**-algebras over some extension **L** of the variety $\mathbf{CSLat}(\bigvee)$ of $\bigvee$-semilattices, gives the category $\mathbf{L}$-$\mathbf{FTop}(T)$ of *fuzzy* topological structures, which incorporates (L, M)*-fuzzy topological spaces* of A. Šostak and T. Kubiak [3] as well as their various modifications.

The resulting category $\mathbf{L}$-$\mathbf{FTop}(T_I)$ has the simple property of subbasic continuity for its morphisms and, therefore, is topological over its ground category $\mathbf{X} \times \prod_{i \in I} \mathbf{LoL}_i$. Moreover, it gives rise to the category $\mathbf{L}$-$\mathbf{FTopSys}(T_I)$ of *topological systems* in the sense of S. Vickers [6], which, in case of completely distributive underlying $\bigvee$-semilattices of **L**, includes the category $\mathbf{L}$-$\mathbf{FTop}(T_I)$ as a full coreflective subcategory. Further restrictions result in the category $\mathbf{L}$-$\mathbf{FTopSys}(T_I)$ being essentially algebraic over its ground category (different from that of $\mathbf{L}$-$\mathbf{FTop}(T_I)$).

By the opinion of the author, the proposed catalg framework has a reasonable balance between universality and fruitfulness of the obtained theory and thus, can provide a good substitute for the above-mentioned poslat approach to topology.

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Department of Mathematics, University of Latvia, Zellu iela 8, LV-1002 Riga, Latvia, E-mail: sergejs.solovjovs@lu.lv