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CATEGORICALLY-ALGEBRAIC TOPOLOGY IN PROGRESS[1]

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Recently in [12; 13; 14; 15] we began a new line of research with an ultimate goal to provide a new setting for fuzzy topological structures. Based on Category Theory and Universal Algebra, the proposed approach is called *categorically-algebraic* (*catalg*), to underline its motivating theories and to distinguish it from the currently dominating *point-set lattice-theoretic* (*poslat*) framework of S. E. Rodabaugh [10]. The motivating idea of the new developments is quite simple. The classical notion of topology is based on the concept of *frame* (complete lattice with finite $\wedge$ distributing across arbitrary $\bigvee$). The above-mentioned theories of S. E. Rodabaugh replace frames with *semi-quantales* ($\bigvee$-*complete semilattices* equipped with a binary operation) claiming to incorporate almost all of the existing approaches to topology. Unfortunately, the notion of, e.g., *closure space* of D. Aerts [1] which has a clear topological flavor is denied a place in the setting since it is based on the concept of *closure semilattice* ($\bigwedge$-semilattice with the singled out bottom element). To face the challenge, the catalg framework replaces the category of semi-quantales with an arbitrary *variety of algebras*. The main advantages of the new setting can be briefly summarized as follows:

- catalg incorporates poslat as a particular subcase;
- catalg properties instead of poslat peculiarities of objects of study are underlined;
- the border between traditional and fuzzy developments is ultimately erased;
- a bridge between different areas of science is established.

Based on these facts, our slogan is that the topological setting of the fuzzy community should be changed from poslat to catalg. Since the desired shift depends on the availability of a sufficiently rich theory developed, currently the catalg approach is being promoted in four (interrelated) directions:

- Topological structures like, e.g., the above-mentioned topological and closure spaces motivated the concept of *catalg space* providing a common framework for many approaches to topology.
- The study on functorial relationships between topological spaces and *topological systems* of S. Vickers [16] (introduced as a common framework for both topological spaces and their underlying algebraic structures – frames) conducted by J. T. Denniston *et al.* [4; 5] and C. Guido [8] as well as *state property systems* of D. Aerts [1] (deemed to serve as the basic mathematical structure in the Geneva-Brussels approach to foundations of physics) gave rise to the notion of *catalg system* which is closely related to catalg space.

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- The theory of natural dualities developed by D. Clark, B. Davey, M. Haviar, *etc.* [2], which provides a machinery for obtaining topological representation theorems for algebraic structures (like, e.g., the famous representations for Boolean algebras and distributive lattices of M. Stone and H. Priestley), induced the notion of *catalg duality*.
- Categorical frameworks for poslat powerset theories of S. E. Rodabaugh [9] (motivated by the classical notions of *image* and *preimage* operators on powersets) gave a stimulus to provide strict foundations for *catalg powerset operators*, highly relied upon in the previous items.

There is, however, still a plenty of work to do. For example, J. T. Denniston *et al.* [3] presented in February 2010 the notion of *interchange system*, motivated by the concept of *predicate transformer* of E. W. Dijkstra [6] (a map between powersets $\mathcal{P}(X)$, $\mathcal{P}(Y)$ induced by a relation $R \subseteq X \times Y$) and the subsequent idea of M. Smyth [11] to view the sets of predicates (subfamilies of powersets) as topologies. The new theory is based on the above-mentioned concept of topological system and the extension of powerset operators to the category of sets (as objects) and lattice-valued relations (as morphisms) done by C. Guido [7]. It is the purpose of the current talk to provide the catalg version of the new developments, exploiting the above-mentioned notion of catalg system and extending catalg powerset theories to the category of algebra-valued relations.

REFERENCES

[1] D. Aerts, E. Colebunders, A. van der Voorde, and B. van Steirteghem. State property systems and closure spaces: a study of categorical equivalence. *Int. J. Theor. Phys.*, 38(1):359–385, 1999.

[2] D. M. Clark and B. A. Davey. *Natural Dualities for the Working Algebraist*, volume 57 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1998.

[3] J. T. Denniston, A. Melton, and S. E. Rodabaugh. Lattice-valued predicate transformers and interchange systems. In P. Cintula, E. P. Klement, and L. N. Stout, editors, *Abstracts of the 31st Linz Seminar on Fuzzy Set Theory*, pages 31–40. Johannes Kepler Universität, Linz, 2010.

[4] J. T. Denniston, A. Melton, and S. E. Rodabaugh. Lattice-valued topological systems. In U. Bodenhofer, B. De Baets, E. P. Klement, and S. Saminger-Platz, editors, *Abstracts of the 30th Linz Seminar on Fuzzy Set Theory*, pages 24–31. Johannes Kepler Universität, Linz, 2009.

[5] J. T. Denniston and S. E. Rodabaugh. Functorial relationships between lattice-valued topology and topological systems. *Quaest. Math.*, 32(2):139–186, 2009.

[6] E. W. Dijkstra. *A Discipline of Programming*. Prentice-Hall: Englewood, 1976.

[7] A. Frascella and C. Guido. Transporting many-valued sets along many-valued relations. *Fuzzy Sets Syst.*, 159(1):1–22, 2008.

[8] C. Guido. Attachment between fuzzy points and fuzzy sets. In U. Bodenhofer, B. De Baets, E. P. Klement, and S. Saminger-Platz, editors, *Abstracts of the 30th Linz Seminar on Fuzzy Set Theory*, pages 52–54. Johannes Kepler Universität, Linz, 2009.

[9] S. E. Rodabaugh. Powerset operator based foundation for point-set lattice-theoretic (poslat) fuzzy set theories and topologies. *Quaest. Math.*, 20(3):463–530, 1997.

[10] S. E. Rodabaugh. Relationship of Algebraic Theories to Powerset Theories and Fuzzy Topological Theories for Lattice-Valued Mathematics. *Int. J. Math. Math. Sci.*, 2007:1–71, 2007.

[11] M. B. Smyth. Power domains and predicate transformers: a topological view. *Lect. Notes Comput. Sci. (Proc. of the 10th Colloq. on Automata, Languages and Programming)*, 154:662–675, 1983.

[12] S. Solovjovs. Categorically-algebraic frameworks for Priestley representation theory. submitted to Proc. of the 79th Workshop on General Algebra, February 12 - 14, 2010, Olomouc, Czech Republic.

[13] S. Solovjovs. Categorically-algebraic dualities. In *Abstracts of Applications of Algebra XIV*. Institute of Mathematics and Computer Science of Jan Długosz University, Częstochowa, 2010.

[14] S. Solovjovs. Composite variety-based topological theories. In E. P. Klement, M. Radko, P. Struk, and E. Drobná, editors, *Abstracts of the 10th Conference on Fuzzy Set Theory and Applications (FSTA 2010)*, pages 116–118. Armed Forces Academy of General M. R. Štefánik in Liptovský Mikuláš, 2010.

[15] S. Solovjovs. Powerset operator foundations for categorically-algebraic fuzzy sets theories. In P. Cintula, E. P. Klement, and L. N. Stout, editors, *Abstracts of the 31st Linz Seminar on Fuzzy Set Theory*, pages 143–151. Johannes Kepler Universität, Linz, 2010.

[16] S. Vickers. *Topology via Logic*. Cambridge University Press, 1989.