Acta Societatis Mathematicae Latviensis
Abstracts of the 8th Latvian Mathematical Conference, April 9–10, 2010, Valmiera, Latvia

SPECTRAL PROBLEM FOR SECOND ORDER FINITE DIFFERENCE OPERATOR WITH THIRD KIND BOUNDARY CONDITIONS[1]

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The solution of the spectral problem for differential equations with boundary condition of third kind $-u''(x) = \lambda^2 u(x), x \in (0,l), u'(0) - \sigma_1 u(0) = 0, u'(l) + \sigma_2 u(l) = 0$, is in following form: $u_n(x) = C_n^{-1}(\lambda_n \cos(\lambda_n x) + \sigma_1 \sin(\lambda_n x))$, $C_n^2 = 0.5(l(\lambda_n^2 + \sigma_1^2) + \frac{\sigma_2(\lambda_n^2+\sigma_1^2)}{\lambda_n^2+\sigma_2^2} + \sigma_1)$, where $(u_n, u_m) = \int_0^1 u_n(x) u_m(x) dx = \delta_{n,m}$ and λ_n are positive roots of the following transcendental equation: $\cot(\lambda_n l) = \frac{\lambda_n}{\sigma_1+\sigma_2} - \frac{\sigma_1\sigma_2}{\lambda_n(\sigma_1+\sigma_2)}, n = \overline{1, N+1}$. The corresponding discrete spectral problem $Av = \mu v$ for finite difference operator on uniform grid of second order approximation with 3-diagonal matrix A of $N+1$ order is in following form: $-\frac{2}{h^2}(v_1 - v_0(1+\sigma_1 h)) = \mu v_0, -\frac{1}{h^2}(v_{j+1} - 2v_j + v_{j-1}) = \mu v_j, j = \overline{1, N-1}, -\frac{2}{h^2}(v_{N-1} - v_N(1+\sigma_2 h)) = v_N$, where μ is eigenvalue and v eigenvector of $N+1$ order with the elements $v_j, h = \frac{L}{N}, x_j = jh, v_j \approx u(x_j), j = \overline{0, N}$. Using the scalar product $[v^1, v^2] = h(\sum_{j=1}^{N-1} v_j^1 v_j^2 + 0.5(v_0^1 v_0^2 + v_N^1 v_N^2))$ one can prove, that the operator A is symmetrical and $[Ay, y] \geq 0$ [1].

The spectral problem have following solution: $v^n = C_n^{-1}(\frac{\sin(p_n h)}{h}\cos(p_n x_j) + \sigma_1 \sin(p_n x_j))$, $j = \overline{0, N}, \mu_n = \frac{4}{h^2}\sin^2(p_n h/2)$, where p_n are the positive roots of the following transcendental equation: $\cot(p_n l) = \frac{\sin^2(p_n h) - h^2\sigma_1\sigma_2}{h(\sigma_1+\sigma_2)\sin(p_n h)}, n = \overline{1, N+1}$. Determine the constants $C_n^2 = [v^n, v^n]$ we have the orthonormal eigenvectors v^n, v^m with the scalar product $[v^n, v^m] = \delta_{n,m}$. The experiments with MATLAB show, that the two last roots p_N, p_{N+1} can not be obtained from transcendental equation. Depending on the parameter $Q = \frac{l\sigma_1\sigma_2}{\sigma_1+\sigma_2}$ we can obtain one ($Q < 1$) or two ($Q \geq 1$) roots from following new transcendental equation: $\coth(p_n l) = \frac{\sinh^2(p_n h) + h^2\sigma_1\sigma_2}{h(\sigma_1+\sigma_2)\sinh(p_n h)}, n = \overline{N, N+1}$. The corresponding eigenvalues and eigenvectors are $\mu_n = \frac{4}{h^2}\cosh^2(p_n h/2)$, $v_j^n = (-1)^j(\frac{\sinh(p_n h)}{h}\cosh(p_n x_j) - \sigma_1 \sinh(p_n x_j)), j = \overline{0, N}$. Then we have the orthonormed eigenvectors v^n, v^m for all $n, m = \overline{1, N+1}$. For $Q = 1$ the eigenvalue is $\mu_N = 4/h^2, (p_N = 0)$ and the corresponding components v_j^N of the orthonormed eigenvector y^N are obtained as the limit ($p_N \to 0$) of the expression $\frac{v_j^N}{C_N}$ in the following form: $v_j^N = (-1)^j(1 - \sigma_1 x_j)\sqrt{6h/(6l + 2\sigma_1^2 L^3 + L\sigma_1^2 h^2 - 6\sigma_1 l^2)}, j = \overline{0, N}$.

REFERENCES

[1] A.A. Samarskij. *Theory of finite difference schemes.* Nauka, Moscow, 1977. (in Russian)

[1] This work was partially supported by ESF research project 2009/0223/1DP/1.1.1.2.0/09/APIA/VIAA/008 and by the Latvian Council of Science research project 09.1572