RESEARCH OF MANY-VALUED MATHEMATICAL STRUCTURES AND THEIR APPLICATION IN MODELING PROCESSES

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1. Introduction

1.1. Fuzzy Sets and Fuzzy Reasoning: a short introduction into history and present state of arts

The comprehension that in the real situations discourses do not always lead either to true or to false statements and that there are many statements between true and false, that is statements of gradual truth was, obviously, not a new idea in science. As well as not a new was the idea, that not all objects of the real world in respect of some property can be classified as "white" or "black" – there can be many objects of different "shadows of gray colors", that is objects having a given property within a certain degree In particular, such idea was discussed already in Aristotle's works. However, in the modern times scientists started to express serious interest to the problem of gradual truth and the related problem of having a property within a certain degree only since the end of the 19th century. It is also worth to note, that first people to be mentioned in this respect are scientists of broad scientific interests, scientists known both as philosophers and researchers in exact sciences. Charles Peirce (1839-1914 USA), known as chemists, philosopher and mathematician, wrote "Logicians have too much neglected the study of vagueness, not suspecting the important part it plays in mathematical thought". Bertrand Arthur William Russel (1872-1970 Great Britain), an outstanding mathematician, logician, writer and philosopher, discussed these problems in his treatise "An introduction to Mathematical Philosophy". In 1937 there was published the monograph "Vagueness: an exercise in logical analysis" by Max Black (1909 Azerbaijan - 1970 USA), a philosopher and a researcher in the field of quantum mechanics. In this monograph the author considered "consistency profiles in order to characterize quantities without clear borders which he called "vague symbols" and discussed the importance of vague symbols for philosophical problems. Probably the first scientist who tried to study the problem of uncertainty from the point of view of mathematical logic was Jan Lukasiewicz (Poland 1878 - Ireland 1956). Lukasiewicz developed a new logical system, in which statements can be not only false or true, but can be also true/false to a certain degree. In his lecture on March 7, 1918 at the Warsaw University he announced that his logical system "is as coherent and self-contained as Aristotle's logic but which is much richer in laws and formulae." This logical system is actually a three-valued case of what is understood by Lukasiewicz logic now. Independently of Lukasiewicz and starting from different premises, another system of many-valued logics was discovered by Emil Post (Russia 1897 - USA 1954).

One more mathematician whose contribution to the "prehistory" of Fuzzy Mathematics, should be mentioned, was Karl Menger (Austria 1902 – USA 1985). Karl Menger suggested to develop a theory in which the relation "an element belongs

to a set" (which is in the basis of the Cantor set theory) is replaced by the probability of an element belonging to a set. In his research he used objects which can be considered as precursors of fuzzy points and which he called hazy sets.
As we tried to show above, the ideas to find a mathematical concept appropriate to describe objects which are not precisely defined as well as to deal with statements for the validity of which one cannot give a monosemantic answer - "yes" or "no", has emerged in the works of many scientists during the first half of the 20th century. However the credits of founding the theory of fuzzy sets (and "inventing" the term "fuzzy set" itself) are given to the professor of the Berkley University L.A. Zadeh. L.A. Zadeh was born in 1921, in Baku, at that time the capital of the Soviet Azerbaijan. Later he moved with his parents to Iran (they were the Iranian citizens) where they lived till 1944. In 1944 Zadeh went for studies to the USA. In 1951 he has received doctor degree in electrical engineering, and in 1963 was elected as the head of the department of electrical engineering at the University of Berkeley. In 1956 he met S. Kleene (1909-1994), an outstanding logician, the author of the famous "Introduction to metamathematics". Friendship with Kleene had a great influence on Zadeh. In particular, this influence showed itself in his idea to use many-valued logics in order to describe the behavior of complex electrical systems. Later this idea has developed into the concept of a fuzzy set and in a series of papers by Zadeh and his followers different aspects of the theory of fuzzy sets and fuzzy logic and related problems were considered. The inception of fuzzy sets by Zadeh did not remain unnoticed: in the next decade there were published several important papers considering fuzzy sets in theoretical mathematics (Topology, Algebra, Measure and Integral Theory, etc) and in applied sciences (Decision Making, System Analysis, etc.)
In the last quarter of the 20th century and at the beginning of the 21st century the number of works where fuzzy sets have been used increased in an avalanche way. There are thousands of works in traditional branches of theoretical mathematics in the context of fuzzy sets: Topology, Algebra, Measure and Integral Theory, Differential Equations, Probability theory and Mathematical Statistics, Mathematical Modeling, etc.; works considering applications of fuzzy sets, fuzzy logics and related mathematical theories in other sciences: Medicine, Biology, Geology, Chemistry, etc.; works devoted to applications of fuzzy sets and rules of fuzzy logics in engineering and industry.

1.2.Research work related to fuzzy sets in Latvia

The first research work in Latvia, where fuzzy sets were used, can be dated by the end of seventies of the previous century. This research was conducted at the Riga Technical University (at those times Riga Polytechnic Institute) by professor A. Borisovs and his PhD students A. Aleksejevs and I. Fjodorovs. They successfully applied fuzzy sets for the study of image processing problems and control-related problems.
Starting with middle 80-ies of the previous century A.Šostak started intense research work in the context of fuzzy sets at the University of Latvia. His first works in the field of fuzzy sets were related to fuzzy topological spaces. Side by side with works on fuzzy topological spaces, in papers by A. Šostak and coauthors fuzzy uniform and fuzzy proximity spaces were considered. Later I. Uļjane started to work in the field of fuzzy topological spaces. In her PhD Thesis (supervisor A.Šostak, defended in 2009) she has developed the theory of fuzzy topologies on many-valued sets (I. Zvina

studied the problem of fuzzification of ideal-based topological spaces and in her PhD Thesis (supervisor A.Šostak, defended in 2010) she developed a valuable theory revealing important relations between fuzzy topologies, locals and lattices. Since the end of the previous century S. Solovyov started to investigate general problems of fuzzy set theory in the framework of the categorical approach. On the basis of this research he has worked out PhD Thesis "On a categorical generalization of a concept of a fuzzy set." (supervisor A.Šostak, defended in 2007). From the beginning of the 21st century intense research work related to measure and integral in the context of fuzzy sets was carried out by Svetlana Asmuss, professor of the University of Latvia. As different from the works of other authors in the field of measure and integral in the context of fuzzy sets, in the works of S. Asmuss and her PhD student V.Ruža (thesis defended in 2012) not only sets are fuzzy, but also the measure and the integral take fuzzy values, which are fuzzy numbers in the Hutton fuzzy real line. Since the end of the previous century the interest of many researchers was drawn to the theory of fuzzy aggregation operators which found many important applications in different areas of applied sciences. Being inspired by urgent reports at the biannual conferences FSTA (Fuzzy Sets: Theory and Applications) and at other international conferences whose schedule includes problems related to fuzzy sets, some young mathematicians from the Department of Mathematics, University of Latvia, took a great interest in the theory of aggregation operators in the context of fuzzy sets and its applications to the real world problems. Most active in this field were J. Lebedinska (supervisor A.Šostak, PhD thesis defended in 2010), O. Grigorenko (supervisor A.Šostak, PhD thesis defended in 2012) and P. Orlovs (supervisor S. Asmuss, the defense of PhD Thesis is planned for 2013.)

Recently important research work in the context of fuzzy sets revived at the Riga Technical University (RTU). As different from the research work done at the University of Latvia, scientists of the RTU are mainly interested in the practical applications of the machinery of fuzzy sets for solving engineering-type problems, and in this sense they continue the traditions of the research started at the RTU in seventies-eighties of the previous century. For example, in his PhD Thesis (supervisor prof. G. Lauks, defended in 2011) J. Jelinskis has developed a new method for projecting telecommunication systems. This method is based on the use of fuzzy sets and principles of fuzzy logics. In his PhD Thesis J. Jelinskis showed also the advantages of his method if compared with the traditional ones which are based on the use of thresholds.

1.3.From fuzzy sets to many-valued mathematical structures.

The machinery based on fuzzy sets and reasoning on the ground of fuzzy logic is successfully used both in the research of the problems of theoretical mathematics and its applications. However at the end of the 20th century and at the beginning of the 21st century the in the process of development of the science the necessity to introduce, side by side with the theory of fuzzy sets and fuzzy logic, a more general concept and to develop the corresponding theory, which would be appropriate to describe and to investigate situations where usual fuzzy sets are powerless. As examples of successful solutions of this problem one should mention L-fuzzy sets and many-valued mathematical structures. As different from ordinary fuzzy sets, whose values are in the unit interval and are restricted by 0 and 1, in case of L-fuzzy sets and many-valued mathematical structures, the range of values can be any complete infinitely distributive lattice, or a cl-monoid, a quantale or even an object of a more general nature. It is important to emphasize that the concept of a many-valued structure is usually a more general concept than the concept of an L-fuzzy set, since a certain kind of fuzziness can be already in the base of a many-valued structure. For example, already in the ground set of many-valued mathematical structure a certain L-fuzzy equality, showing to what extent two given elements of the set are equal, can be introduced. In a natural way we can speak also about the synthesis of the both concepts that is about L-fuzzy many-valued mathematical structures. Thus, from one side, the scope of possible applications of L-fuzzy and many-valued mathematical structures is much wider that the scope of ordinary fuzzy sets, while on the other the mathematical problems related to the study of L-fuzzy and many-valued mathematical structures become essentially more complicated. Besides, the additional difficulties of research are caused by the fact that when investigated the problem now one must take into account which object is taken as the range for L-fuzzy and many-valued structures. It is just the investigation of different L-fuzzy, many-valued and L-fuzzy many-valued mathematical structures are the main goals of the work of our group in this Project. More about this goal we shall expose in the next subsection. At this moment we would like just to note that the large experience of the members of our group described in the previous subsection testifies of a good research potential and serves as the guarantee of the successful accomplishment for the tasks of the project.

1.4. The main goals of the activities of the working group of the quantum ”Many-valued mathematical structures and their applications in modeling of the processes”

The principal goal of the quantum group in this Project is to investigate actual mathematical problems connected with L-fuzzy, many-valued and L-fuzzy many-valued mathematical structures. Specifically our task was to research topological, algebraic and analytic many-valued mathematical structures, and on the basis of the obtained results to develop further the theories of L-fuzzy, many-valued and L-fuzzy many-valued mathematical structures. In the framework of these theories it was planned to construct L-valued categories of different (L-fuzzy) many-valued mathematical structures and to study basic properties of these L-valued categories. It was planned to work out the concept of an aggregation operator, in particular, of a measure and an integral in the context of L-fuzzy and many-valued mathematical structures. The development of a valuable theory of approximate systems in the context of (L-fuzzy) many-valued structures was one of the principal goals of the work of the quantum group. Basing on the obtained results in the framework of this theory it was envisaged to describe and to study approximate schemes applicable for the case of imprecise, fuzzy or many-valued information.

2. The survey of the main results obtained in the project

We describe the main results obtained in the process of realization of the Project following the structure according to the principal issues which were investigates in the project.

2.1. Fundamental Issues of the Research in the Area of Many-valued Mathematical Structures.

The main goal for the research of this issue of the project was to create a unified approach to lattice-valued topology, which would provide a common framework for the majority of the existing topological settings. We call the new theory *categorically-algebraic topology* (*catalg topology*, for short), to underline its dependence on category theory and universal algebra. Additionally, the term "categorically-algebraic" points out the difference between our setting and the currently popular *point-set lattice-theoretic topology* (*poslat topology*, for short), which provided a partial motivation for our introduced notions and obtained results. One of the main advantages of catalg topology is the fact that the latter erases the border between fuzzy and crisp approaches, propagating algebra as the main driving force of modern exact sciences.

Our participation in the research project enabled us to formulate the axiomatic foundations of catalg topology, and to develop the basics of its theory. Our approach follows the methods of algebraic theories of F. W. Lawvere, which currently form the backbone of categorical algebra. More precisely, we introduced a general notion of topological theory and the category of its models. The topological theories themselves, however, form a category, i.e., there exists a convenient and fruitful notion of morphism of topological theories. These morphisms induce functors between the categories of models of topological theories in question, thereby providing a way to shift the results from one topological setting to another. Similar to the ideas of categorical algebra, the main point of catalg topology is the study of the category of topological theories. It soon appeared that the notion of topological theory is heavily dependent on the so-called *powerset theories*. More precisely, the classical topology is based in the concepts of power-set of a set, as well as of backward powerset operator induced by a map. Their respective generalizations for the above-mentioned poslat topology motivated us to introduce the notion of catalg powerset theory, considering both backward and forward ones. In particular, we provided rigid categorical foundations for both catalg powerset theories, employing the notion of quantaloid, which recently has been announced to make a cornerstone of modern lattice-valued mathematics.

Having introduced the foundations of catalg topology, we turned our attention to several applications of the new theory. One of the most important of them appeared to be a generalization of *topological systems* of S. Vickers. The latter concept was brought to life by its initiator with a view to obtain a common framework for both topological spaces and their underlying algebraic structures – locales. More precisely, the category of topological systems contains the category of topological spaces and the category of locales as a coreflective and a reflective subcategory, respectively. The (co)reflection property comes from the existence of the so-called *spatialization* and *localification* machineries, which provide a topological space and a locale from a topological system, respectively. Several fuzzy topologists tried to lift the results of S. Vickers to L-fuzzy topology, failing though to overcome the difficulties, while accommodating L-fuzzy topology inside crisp topological systems. With our new catalg approach in hand, we provided the concept of catalg topological system, which allowed us to restore both the spatialization and the localification procedures in the catalg setting.

The concept of catalg topological system motivated a bunch of different applications of its respective theory. First of all, we found out that the concept incorporates the well-known notion of *state property system*, which serves as the basic mathematical structure in the Geneva – Brussels approach to foundations of physics. In particular, we showed that the well-known correspondence between state property systems and closure spaces is a particular instance of the above-mentioned catalg spatialization procedure for topological systems. Secondly, it appeared that the notion of *attachment relation* between fuzzy points and fuzzy sets, introduced by the Salento research group (Italy) as a fuzzification of the "element of" relation between points and sets, has a direct relation to catalg topological systems. More precisely, we introduced the notion of catalg attachment relation, and showed that the latter concept gives rise to several morphisms between certain topological theories; one of their corresponding functors between the categories of models is the important *hypergraph functor*. This aspect of the theory of catalg topological systems was thoroughly studied. Thirdly, the theory of catalg topological systems turned out to be much related to the theory of *Formal Concept Analysis* (FCA), introduced by German mathematicians at the end of the last century, as a new area of research, which is based in a set-theoretical model for concepts and conceptual hierarchies. In the process of realization of the Project we introduced a new framework for lattice-valued FCA, which was based in catalg topological systems. In particular, we made the first steps in the catalg approach to the theory of *Galois connections*, which goes back up to 1944 (a paper of O. Ore) at least.

Additionally, the concept of catalg topological space itself motivated two fruitful developments of its theory. On the one hand, we considered the relationships between fuzzy topology and *non-commutative topology* (developed either in the framework of C^*-algebras, or recently in the framework of quantales, and motivated by the wish to extend the duality of I. Gelfand and M. Neumark between the categories of Hausdorff locally compact topological spaces and commutative C^*-algebras to non-commutative C^*-algebras) in the framework of catalg topology. Among other things, it appeared that the theory of non-commutative topology incorporates the theory of many-valued topology, which clarified the link between quantumness and fuzziness. Secondly, we found a plenty of applications for catalg topology in the theory of *natural dualities*, which took its inspiration in the famous topological representation theorems for abelian groups, Boolean algebras and distributive lattices of L. Pontrjagin, M. H. Stone and H. Priestley, respectively. These representation theorems opened a way to transform algebraic problems, stated in an abstract symbolic language into topological problems, where geometric intuition comes to our help. With the catalg topology in hand, we provided catalg analogues of natural dualities, thereby paving the way for applications of lattice-valued topology in topological representations of algebraic structures. In particular, we considered a catalg version of the famous adjunction between the categories of topological spaces and locales (due to D. Papert and S. Papert), which resulted in the convenient notions of catalg sobriety (for topological spaces) and spatiality (for locales). The adjunction in question was studied by us in the form of both fixed-basis and variable-basis approach, motivated by the corresponding notions of lattice-valued topology.

Since catalg topology relies heavily on universal algebra, the study of those algebraic structures, which are closely related to topology, is much desirable to the development

of the whole new theory. With this idea in mind, we considered the notion of *extended-order algebra*, which is a common setting for the majority of algebraic structures employed in mathematics of many-valued structures. In particular, we presented different versions of extended-order algebra morphisms, studied their respective categories, and showed that under suitable requirements on their objects, the latter share the most important properties of the much studied category of partially-ordered sets and order-preserving maps.
Towards the end of our participation in the project, we turned our attention to the relationships between catalg topology and *universal topology*, which is a particular branch of *categorical topology* (the latter mostly concerned with the study of topological categories and their relationships to each other), motivated by the theory of universal algebra. In particular, we showed that a concrete category is fibre-small and topological if and only if it is concretely isomorphic to a subcategory of a category of catalg topological structures, which is definable by topological co-axioms in it. This important result essentially says that the whole theory of fibre-small topological categories (including, therefore, lattice-valued topology) can be done in the setting of catalg topology. Moreover, the achievement in question enabled us to introduce a general fuzzification procedure for (fibre-small) topological categories, a particular instance of which appeared to be the theory of (L,M)-fuzzy topology.

2.2. Many-valued topological structures

The research of fuzzy topological spaces in Latvia was initiated in 80-ies of the previous century in A.Šostak's works. In these works the essential part of the theory of fuzzy topological spaces was developed. However, recently the problem to study just L-fuzzy and, more generally, many-valued topological structures and to develop the corresponding theory of many-valued topological spaces became actual, Besides the problem to consider topological type structures both with variable base and variable range became also of importance. Side by side with working out the theory of many-valued topological structures, the problem to develop the corresponding theories of many-valued proximities, many valued uniformities and generally many-valued syntopogenous structures in accordance with many-valued topologies became actual. Just these problems were studied in the Project and the main results obtained in this area are discussed in this subsection.
The machinery worked out on the basis of soft sets is applied for the study of categories of many-valued topological spaces. In particular, in the framework of the category of soft sets the categories of L-fuzzy topological spaces, L-valued topological spaces, many-valued topological spaces and (L,M)-fuzzy topological spaces are characterized; here L is an arbitrary complete infinitely distributive lattice and M a complete completely distributive lattice. The research machinery based on soft sets allowed to obtaine new information about these categories. In particular, it was used for construction of final and initial structures for families of mappings in these categories.
A functor assigning to an M-valued L-fuzzy topological space an Idl(M)-valued topological space, where L is an infinitely disgtributive complete lattice, M is a completely distributive lattice and Idl(M) is the complete lattice of all ideals of the lattice M. The construction of this functor is used to investigate the complete lattice formed by all (L,M)-fuzzy topologies an M-valued L-topological spaces on a given set.

The concept of a syntopogenous structure and of a syntopogenous space where introduced in 1963 by a Hungarian mathematician A. Csaszar. Later in the works by Csaszar and other mathematicians a well-grounded theory of syntopogenous structures was developed; this theory has found important applications in the research of categories of topological, uniform and proximity spaces. Recently, reflecting the increasing role of many-valued mathematical structures and the interest in this subject of many scientists, the challenge to develop many-valued versions of a syntopogenous structure and of a syntopogenous space was both quite natural and pressing. Developing this field in the framework of the Project the concepts of a many-valued syntopogenous structure and of an L-fuzzy syntopogenous structure as well as the corresponding many-valued syntopogenous spaces and L-fuzzy syntopogenous spaces were introduced and studied. In this research L is a complete infinitely distributive lattice with an implicator, which specifically can be introduced by means of a lower semi-continuous t-norm on the lattice L. Mappings between many-valued and L-fuzzy syntopogenous are considered and the property of continuity of such mappings is introduced and studied. Basing on many-valued and L-fuzzy syntopogenous spaces as objects and their continuous mappings as morphisms the category SYNT(L) is constructed. Basic properties of the category SYNT(L) are studied; the operations of product and coproduct as well as the operations of taking quotients and subobjects of many-valued and L-fuzzy syntopogenous spaces are efficiently defined and studied. The category of many-valued topological spaces and continuous mappings, the category of many-valued uniform spaces and uniformly continuous mappings, and the category of many-valued proximity spaces and proximally continuous mappings are characterized and investigated in the framework of the category SYNT(L).
New results related to the so called Namioka's problem were obtained. The essence of this problem is to characterize those sets of points in the product of two topological spaces at which a partially continuous function (that is a function, which is separately continuous at each of the coordinates) is jointly continuous. These results were got by means of the so called topological game machinery. Besides the progress in the study of this problem in the classical, that is a crisp version, some results related to the many-valued version of this problem were obtained. Note that earlier, as far as we know, the fuzzy and many-valued versions of the Namioka problem were not considered.
The study of the use of many-valued, in particular, fuzzy metrics and pseudometrics in the research of problems related to digital topology was initiated. The results of this research can have applications in theoretical computer science and it is planned to continue the work in this direction also after the Project is completed.

2.3. Many-valued bornologies

Bounded sets and the concept of boundedness of a function, which reflecting this property, is one of the most important conceptions both in the theory of metric and pseudometric spaces and in the theory of topological vector spaces. Aiming to extend the concept of boundedness to the case of mappings between arbitrary topological spaces, an Australian mathematician of Chinese descent H.S. Hu introduced in 1943 the concept of bornology applicable to any topological space and of the corresponding concept of a bornological space. Later H.S. Hu and co-authors developed this concept into a valuable theory of bornological spaces. One of the principal goals of the Project was to work out the conception of a many-valued bornology and to develop the corresponding theory of many-valued bornological spaces.

In the course of the realizations of the Project two closely related but nevertheless different approaches to the concept of bornology in many-valued setting were worked out. The first one of these approaches, is the concept of an L-fuzzy bornology on a set is based on an ideal of L-fuzzy subsets of a given set, the members of which cover this set. Among the main problems studied in the frames of this approach are the description and the research of the lattice all properties of L-fuzzy bornologies on a given set. Another problem considered in this context is the construction of the corresponding category of L-fuzzy bornological spaces. In order to develop this approach the concept of a bounded mapping between two L-fuzzy bornological spaces is introduced and the corresponding category is defined. L-fuzzy bornological spaces and bounded mappings between L-fuzzy bornological spaces make a category which is studied in the frames of the Project. In particular, final and initial L-fuzzy bornological structures for families of bounded mappings of L-fuzzy bornological spaces are constructed. Initial a final L-fuzzy bornological structures allowed us to work out the efficient construction of the most important structures of L-fuzzy bornological structures, such as products, coproducts, quotients, characterizing subobjects, etc.
The second, alternative approach to the problem of fuzzification of the concept of bornology is based on interpreting an L-valued bornology as a mapping defined on the exponent of a given set, taking values in the lattice L and satisfying certain properties, which are L-valued analogous of the axioms used in the definition of a crisp bornology. The value in L which the L-fuzzy bornology assigns to a given subset of the given set is realized the degree to which this subset is bounded. A set X with the corresponding L-valued bornology defines an L-valued bornological space. After introducing the concept of boundedness of mappings between L-valued bornological spaces, we come to the category of L-valued bornological spaces and their bounded mappings. Basic properties of this category are studied. Among other it is proved that this category is topological over the category SET of sets. Basic properties of this category are studied. In particular, initial and final L-valued bornologies in this category are constructed and studied.
A method allowing to construct an L-valued bornology from a chain of ordinary bornologies is developed. Special properties of this construction are studied. Among other this construction is applied to construct L-valued bornologies from families of fuzzy metrics and fuzzy pseudometrics. The basic properties of the L-valued bornological spaces obtained from a chain of fuzzy metrics and fuzzy pseudometrics are investigated.

2.4. Approximate systems

As it was already noticed in the previous items, in the last two decades different approaches to the concepts of an L-fuzzy topology and many-valued topology were discussed in the literature and the basics of the corresponding theories were worked out. Therefore it was both natural and urgent to find a unified approach to all these theories, in which, specifically the theory developed in our Project and discussed in item 2.1 should be taken into account.
On the other hand side by side with the fast development of theories of L-fuzzy and many-valued topologies as well as their applications one can take note of a fast increasing interest to the so called rough sets and their extension as fuzzy rough sets among scientists working in different areas where mathematical methods are used.. In this concern we recall that the concept of a rough set introduced by Polish scientist Z.

Pawlak in 1983 recently has draw attention of specialists in different areas in particular due to the possibilities which they present in the process of converting of large amount of information with minimizing the expenditures. It could seem unexpected, but in fact one can notice something essential which is in common both for all approached in L-fuzzy and many-valued topologies on one side and different concepts related to rough sets, in particular, many-valued rough sets. So, it was both as a challenge and as a timely task, to develop an approach and the corresponding theory which would cover many-valued topological issues as well as rough-set related issues and be applicable for their research. To solve this problem in the process of working out the project the concept of an approximate system, specifically of an M-approximate system where M is a complete infinitely distributive lattice enriched with a commutative associative binary operation was worked out, the basics of the corresponding theory were developed and the possible applications of this theory were investigated.

In the framework of the Project the theory of M-approximate systems was used to investigate rough sets, L-fuzzy rough sets and also to introduce and to study many-valued rough sets. The concept of an M-approximate system on a complete infinitely distributive lattice L is based on lower and upper M-approximate operators which are defined on a the lattice L and take their values in the lattice M. In the process of realization of the Project a general theory of M-approximate systems was worked out. Special types of M-approximate systems are introduced: lower semi-continuous from above and upper semi-continuous from above M-approximate systems, involutive M-approximate systems. The properties of such M-approximate systems are carefully investigated. This type of M-approximate systems are important both for the study of many-valued topological structures and fuzzy rough sets. M-approximate systems with a fixed ground lattice L are studied. For the research of such systems a method of topologization of an M-approximate system worked out in the Project. Applying this method it is proved that the family of all M-approximate systems with a fixed ground lattice L is an infinitely distributive lattice; the properties of this lattice are studied.
Morphisms between M-approximate systems are constructed both in case when the ground lattice L is fixed and when it can vary. M-approximate systems and their morphisms form a category AS(M). Basic properties of this category are investigated. For families of morphisms in category AS(M) initial and final structures are constructed and studied. In its turn these constructions allow to work with product and coproduct operations, as well as with the operation of taking a quotient in category AS(M). It is proved that category AS(M) is topological over the category of infinitely distributive complete lattices. It is proved that the categories USC-AS(M), LSC-AS(M) and I-AS(M) formed respectively by upper semi-continuous from above, lower-semi-continuous from above and involutive M-approximate systems are reflective and coreflective subcategories of the category AS(M). Further properties of these subcategories are studied. In particular, initial and final structures for families of morphims in these subcategories are constructed.
One of the main goals for introducing and investigating M-approximate operators and M-approximate systems is the good prospects of these concepts for the construction and the study of different topological-type objects. In the process of realization of the project categories of many-valued, or (L,M)-fuzzy topological spaces, Chang-Goguen L-fuzzy topological spaces, Hutton fuzzy topological spaces, Rodabaugh variable base L-fuzzy topological spaces are characterized and studied in the framework of the category AS(M) of M-approximate systems. It is important to note that the use of M-

approximate systems and the corresponding theory worked out in the project allows to use a unified approach to the study of all these categories. In the long run this approach is based on interior and closure topological-type operators which are realized as the lower and the upper M-approximate operators, respectively.
Side by side with the important role of M-approximate operators and M-approximate systems in the study of many-valued topological spaces discussed in the previous paragraphs, M-approximate systems and M-approximate operators have good prospects in the study and the use of rough sets. In the framework of the Project the theory of M-approximate systems was used to investigate rough sets, L-fuzzy rough sets and also to introduce and to study many-valued rough sets. When applying M-approximate systems for the study of rough set-type structures lower and upper M-approximate operators were interpreted respectively as an inner and outer set-approximations.
In the process of studying real-world problems one encounters often to necessity of defuzzification of the result. By defuzzification we mean the process of replacing of an L-fuzzy or many-valued object obtained in the process of the research by a crisp object which in a certain sense is the best way approximates the fuzzy or many-valued object. In the process of realization the Project it was studied how M-approximate systems can be applied for solving defuzzification problems. The obtained results were testified by real-world problems.
The study of approximate systems with a variable range M was initiated in the framework of the Project. Thus the situation when not only the domain of an approximate system is many-valued and can vary, but also its range is many-valued and is subject to vary. Such approximate system generalization, in particular, well correlate with categorical-algebraic topologies and topological systems with a varied base discussed in item 2.1.. In the Project we have studied the lattice properties of the family of approximate systems with variable range M. Morphisms between variable range approximate systems are introduced, and the corresponding category AS of approximate system is constructed. The properties of this category are carefully investigated. For families of morphisms in this category are constructed initial and final structures: on the basis of these structures the operations of product, coproduct, taking a quotient object and a subobject in this category are efficiently described. It is proved that the category AS is topological over the category of infinitely distributive complete lattices. The category AS(M) with a fixed lattice M discussed in the previous paragraphs is proved to behave as a reflexive and coreflexive subcategory of the category AS. Special types of approximate systems with variable base are extracted and studied, in particular, such as lower and upper semi-continuous from above approximate systems and involutive approximate systems.

2.5. Approximative schemas

In the literature one can find two different approaches to the approximation problem when the conditions which must be satisfied under the process of approximation are not strictly defined. One of them, we call it the statistical approach, is applicable in cases when the approximation relies on information given with some error and is based on the use of distribution functions. The second, so called interval method considers approximation when imprecision of the used information is implied by inevitable interval estimation. As different from these two approaches we describe an alternative approach to approximation problem which could be helpful in cases when the imprecision of information used in the approximation process has fuzzy or vague

nature. Our research in this direction is devoted to methods of approximation under L-fuzzy information, where L is a complete, completely distributive lattice.
In order to estimate the "quality" of approximation under L-fuzzy data, we need an appropriate *L*-fuzzy analogue of a real number. Notice that the error of a method of approximation in this case is characterized by the supremum of the corresponding L-fuzzy set of real numbers. The principal aim of our work is to generalize for the case of L-fuzzy information the idea of an optimal error method of approximation (the method whose error is the infimum of the errors of all methods for a given problem using the same information) and the concept of a central algorithm, which is always an optimal error algorithm and in the crisp case is useful in practice as well as in the general theory. In the classical approximation theory central methods for solution of linear problems with balanced information usually are based on splines. In the crisp case one can distinguish three types of splines used in the study of approximation problems: they are respectively the interpolation splines, the smoothing splines and, the most general, splines on the basis of convex sets. Generalizing the last type of crisp splines, here we introduce the concept of an L-fuzzy spline on the basis of L-fuzzy sets and with the help of such splines initiate to develop an approach to study of the approximation problem in case when L-fuzzy information is taken as the basis for this approximation.
Our research deals with the generalized mixed interpolating smoothing problem in the abstract setting of a Hilbert space. The main aim of this work is the development of the general theory of the space of mixed interpolating-smoothing splines. This space gives the solutions of the minimization problems with combined smoothing and interpolating conditions. The generalized problem involves particular cases such as an interpolating problem, a smoothing problem with weights, a smoothing problem with obstacles, a problem on splines in convex sets and so on. The problems of characterization of the splines from the generalized space are solved. The questions of the existence and unique solution of the interpolating-smoothing problems are investigated. The theorem on existence of a solution of the problem is proved in the general case and characterization of a solution of this problem for particular cases is considered. Two problems of approximation in Hilbert spaces are considered with one additional equality condition: the smoothing problem with a weight and the smoothing problem with an obstacle. This condition is a generalization of the equality, which appears in the problem of approximation of a histogram in a natural way. We characterize the solutions of these smoothing problems and investigate the connection between them.
During the last three decades the problem of approximation of the density function of a random value by splines has been investigated in the different statements and under different restrictions on its solutions. This problem is the problem of nonparametric estimation of a density function and its formulation is still being topical. Firstly for the solution of this problem interpolation splines were used, later smoothing splines were considered to be more useful. In our research smoothing splines are investigated under one additional condition, which we consider to be very important, as it corresponds to the property of density histogram (the area under it is equal to 1). We consider the problem of approximation of a density histogram in two different statements: the problem of smoothing histopolation with a weight parameter and the problem of smoothing histopolation with an obstacle and obtain the connection between their solutions. This connection allows us in order to solve the problem with an obstacle, to reduce it to the problem with a weight parameter, and using the

algorithm of solving the easier problem to find the solution of a problem with an obstacle.
We develop methods of F–transforms (fuzzy transforms) based on splines. The core idea of F–transforms proposed by I. Perfilieva is based on a fuzzy partition of an interval into fuzzy subsets (fuzzy partitioning), determined by their membership functions. In this work we consider polynomial splines with defect 1 as membership functions of two types of fuzzy partition: classical and generalized. The idea of F–transform is transformation from a function space to a finite dimensional vector space, the inverse F–transform is transformation back to the function space. We claim that for a sufficient representation of a function we may consider its average values over fuzzy subsets from the used partition. We consider also F–transform of higher degree with respect to both types of fuzzy partitions. Error bounds for both transforms are obtained and analyzed for a given function and its derivatives, as well as for classes of functions. We investigate approximative properties of the spline based inverse F–transforms and illustrate them with numerical examples.

2.6. L-fuzzy measure and integral

In the context of L-sets (in case L is a complete, completely distributive lattice with the minimum t-norm) we develop the theory of measure and integral. The main purpose of our research in this direction is to introduce the concept of measure and integral taking values in the L-fuzzy real line. We suggest the construction of an L-fuzzy valued measure by extending a measure defined on a σ-algebra of crisp sets to an L-fuzzy valued measure defined on a tribe of L-sets. We introduce an L-fuzzy valued integral of a real valued function over an L-set with respect to an L-fuzzy valued measure, consider its properties and describe methods of L-fuzzy valued integration. In a similar way to the classical case we define a fuzzy valued integral stepwise, first considering the case of characteristic functions then extending it for simple non-negative measurable functions and finally for non-negative measurable functions. We describe and investigate integrable over an L-set with respect to an L-fuzzy valued measure functions.
We show some possible applications of an L-fuzzy valued integral in the approximation theory. For problems that can be solved only approximately the notion of the error of a method of approximation plays the fundamental role. In order to estimate the quality of approximation on an L-fuzzy set, we need an appropriate L-fuzzy analogue of a norm. We introduce an L-fuzzy valued norm defined by an L-fuzzy valued integral with respect to an L-fuzzy valued measure. We describe the space of L-fuzzy integrable real valued functions. We also show how the introduced L-fuzzy valued norm can be applied to estimate on an L-set the error of approximation for a given real valued function, as well as for classes of functions. The obtained results are illustrated by numerical examples.
We present a methodology of decision making on approximation methods under uncertainty given by L-fuzzy sets. We consider approximation, when a set we approximate over is L-fuzzy, meaning that our interest is more focused at some parts of the set. In this context for approximation methods we discuss criteria of the error optimal decision.

2.7. L-fuzzy aggregations

Our investigations are devoted to a general aggregation operator acting on L-fuzzy real numbers, where L is a complete, completely distributive lattice. General aggregation principles are described by using a T-extension of an ordinary aggregation operator based on a t-norm T. Factoraggregation operators are introduced and investigated. The aim of our research is to analyze properties of the general aggregation operator and general factoraggregation operator depending on properties of the ordinary aggregation operator and the t-norm. By using the general aggregation operator we describe some t-norm based operations with L-fuzzy real numbers and investigate their properties. We consider some possible applications of factoraggregations in multilevel linear programming and theory of fuzzy games.
By using factoraggregation constructions we consider bi-level linear programming problems with a single objective function on the upper level and multiple objective functions on the lower level. In order to obtain an appropriate compromise solution M. Sakawa and I. Nishizaki have introduced special parameters, described by membership functions of the objectives. Our research is devoted to an analysis of these parameters by using the specially designed general aggregation of the lower level objective functions considering the classes of equivalence generated by the upper level objective. The method is illustrated by numerical and graphical examples.
Factoraggregations are applied to an analysis of fuzzy games. The classical game theory assumes that all data of a game are known by players. However, in real game situations often the players are not able to evaluate exactly some data of the game. It means that the certainty assumption is not realistic in many occasions. This lack of precision may be modelled by different ways and a fuzzy approach is one of them. Our investigations deal with non cooperative two-person games with fuzzy pay-offs. Namely, we consider two-person zero-sum games with fuzzy pay-offs (matrix games where each component of the pay-off matrix is a fuzzy number). We describe a method of investigation of such games by finding equivalent multi-objective linear programming problems whose solutions give values of the games. This method provides the possibility to solve the value of the game as a fuzzy number. Fuzzy linear programming problems are transformed to the linear programming problems by using a grid of scale. We describe the formal definition of the value of a fuzzy pay-off matrix game and develop a fuzzy programming method to evaluate it by solving the corresponding multi-level linear programming problem. The validity and applicability of the proposed methodology are illustrated by numerical examples. Finally, the obtained values are compared with the results obtained by means of the most important models introduced by L. Campos, D.F. Li and J. B. Yang, C.R. Bector and R.R. Yager, T. Maeda.

2.8. Many-valued orderings

The objectives of research in this direction are many-valued order relations; the development of the theory of monotonicity degree for mappings of ordinary sets L-fuzzy sets and many-valued sets; construction of the L-valued category in accordance with this theory; application of the obtained results and constructions in the study of aggregation processes.
In the process of realization of the Project the prospects of the use of the theory of many-valued orderings in the modeling of aggregation of information were studied and the advantages of this theory if compared with the traditional ones are investigated. In the process of development the theory of many-valued ordering the concept of an L-fuzzy monotonicity was introduced; this concept is essentially a fuzzy

version of the property of ordinary monotonicity and it can be successfully applied for the study of problems dealing with different types of aggregation. Based on the use of fuzzy relations the degree of monotonicity for a mapping is introduced. According to this concept every mapping is monotone to a certain degree as different from the classical case where each function is either monotone or not monotone. One of important advantages of the new concept is that it is appropriate for working with large data bases allowing to get free of a small corruption in the data.

The concept of an ordering is extended to the concept of an many-valued ordering both for a usual set, for an L-fuzzy set and for an L-valued set; in the process of studying these concepts an auxiliary fuzzy relation, so called weak many-valued ordering relation plays a crucial role. Among other, these ordering relations are analyzed from the categorical point of view and L-valued categories appropriate for the study of these relations are constructed. Special objects and special morphisms of these categories (like sections, retractions, isomorphisms, epimorphisms etc) are investigated.

The perspectives of the use of many-valued ordering relations for solving problems of linear programming are investigated. The results are successfully applied for the study of multi-objective linear programming problems. The results obtained theoretically are testified by computer programmes Maple and R-project.

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