RESEARCH OF DIFFERENCE EQUATIONS RELEVANT TO APPLICATIONS

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Introduction

Goals and Objectives

A major research goal has been establishing novel sufficient conditions for equivalence of systems of difference equations in a neighbourhood of invariant manifolds both in spaces of finite dimension and general Banach spaces. This contributes to the qualitative and quantitative research of application motivated systems of difference equations and continuous and discrete dynamical systems including time-scale systems.

Among other research objectives was simplification and linearization in the neighbourhood of an invariant manifold of various kinds of difference equations (including trichotomic) in finite dimension spaces as well as in more arbitrary Banach spaces using Green mappings. Difference equations of such kind are obtained from irreversible partial differential equations of evolution widely using in modelling physical processes. The reduction principle developed by the group leader was to be applied. Further development of stability theory and equivalence theory of dynamic systems was planned in the context of time scales, which constitute a theoretical basis for unification of difference equations and differential equations. The results obtained by the Riga school of nonlinear boundary value problems for differential equations might thus be transferred to the field of difference equations. Work on problems of difference equations should help identify fertile research topics for the young scientists.

A detailed outline of the study

1. Research on Difference Equations and Associated Dynamic Systems.

Because of the prospective applications become topical Difference equation studies. Based on the internationally accepted qualitative theory of differential equations different aspects of the project which aims to find new sets of sufficient conditions for the system of equations Difference invariant varieties around the final dimensions, it would be equivalent to the Banach space, develop Difference and associated dynamical systems theory solution stability and dynamic of equivalence theory, as well as qualitative and quantitative research in various aspects of continuous and discrete dynamic systems including the "timeline" ("time scale") system, which combines a unified theory of differential equations and difference equations.

Stability theory and the principle of reduction commitments actuality discussed in [10, 12] and presentations at conferences Valmiera (2012) and Dresden (Germany) (2010). If the subsystem is a key part and the corresponding linear matrix eigenvalue real parts are positive, but non-linear part satisfies the Lipchitz conditions small enough Lipchitz constant, then the trivial solution stability study was reduced to a lower dimension analogue system, which the linear part of the matrix eigenvalue real parts are equal to zero, but the nonlinear part satisfies the Lipchitz conditions of low Lipchitz constant. In order to prove first prove the so-called centre manifold existence theorem. To prove this claim in waste solution to the integral representation of the so-called constant variation formula generalization. The following proves that every output system solution tended to one particular solution to the centre manifold, the so-called asymptotic phase-type theorem. In this demonstration Functional use of specific methods and evaluation techniques. At times, it is noted that the centre-type manifolds and the asymptotic phase property can be shown to the more general terms, the so-called separation conditions to be looked at in detail in the given presentations. Based on the above theorems can be proved the principle of reducing stability theory, the system outputs the trivial solution is stable, asymptotically stable and unstable Lyapunov sense if and only if the reduced equation trivial solution is stable, asymptotically stable or unstable Lyapunov sense. Next we consider the more general case, i.e. the main part of the system is essential to the so-called nonlinear Florio-Seibert problems, paying special attention to equations with homogeneous principal part at a sufficiently general conditions show closed, asymptotically stable invariant sets (attractors) Having used the existence of global Lyapunov-Krasovsky functions. At times, it is noted that the invariant set is not necessarily invariant variety. Find sufficient conditions through Ļyapunov-Krasovsky function at which the invariant set is Lipchitz variety. Next find other sufficient conditions for the existence of invariant varieties. Is dealt one parameter family of mutually commuting operators. It turns out that this family can prove the existence of a common fixed point. The proof is based on the original and Sauder-Lerje theorems principles. In general, not fulfilled asymptotic phase characteristic of homogeneous equations to the main part. This is illustrated by a specific example. Are found sufficient conditions of a fundamental nonlinear systems trivial solution stability studies of reduced system is equivalent to the trivial solution existence.

Works [16,19,22,24] and presentations at conferences Ariel (Israel) (2010), Moscow (Petrovsky seminar) (2011), Loughborough (UK EQUADIFF) (2011) and Batumi (Georgia) (2011) studied the so-called impulsive dynamical systems in general Banach functional space, including the question of Lipchitz smooth invariant varieties of the existence of a solution to the asymptotic solution to the exponential phase and the desire for a stable invariant variety, as well as reducing the principles of stability theory, the dynamic equivalence theory. Note that the impulsive dynamical systems characteristic is that they are often unedged. So we look at the general impulsive dynamical systems Banach space separating the linear and nonlinear part., As well as the distribution of the dynamic system and jump conditions

$$\frac{dx}{dt} = A(t)x + f(t,x,y),$$

$$\frac{dy}{dt} = B(t)y + g(t,x,y),$$

$$\Delta x|_{t=\tau_i} = x(\tau_i + 0) - x(\tau_i - 0) = C_i x(\tau_i - 0) + p_i(x(\tau_i - 0, y(\tau_i - 0)),$$

$$\Delta y|_{t=\tau_i} = y(\tau_i + 0) - y(\tau_i - 0) = D_i y(\tau_i - 0) + q_i(x(\tau_i - 0, y(\tau_i - 0)) .$$

Assuming that the linear part of the evolution operator satisfies the integral of the separation conditions and non-linear conditions Lipchitz members meet with a small constant, proving the existence of Lipchitz separable manifold. The following shows that solutions to satisfy some type integrated inequalities. Based on the results and viewing the specific type of functional integral, proving asymptotic properties of the phase. All this allows to formulate and prove the principle of reducing the integral version. Impulsive system trivial solution is integrally stable integral asymptotically stable or unstable integral if and only if the system of reduced impulse

$$\frac{dx}{dt} = A(t)x + f(t, x, u(t, x)),$$
$$\Delta x|_{t=\tau_i} = x(\tau_i + 0) - x(\tau_i - 0) = C_i x(\tau_i - 0) + p_i(x(\tau_i - 0), u(\tau_i - 0, x(\tau_i - 0))),$$

trivial solution is integrally stable integral asymptotically stable or unstable integral. In case invariant variety is exponentially asymptotically stable, the more accurate the result is correct, namely the trivial solution is stable, asymptotically stable or unstable as the output of dynamic systems, the Normalized dynamic system.

The major studies of various types, including the type of swing trihotomisk impulsive dynamical systems to simplify and linearising around a stationary point as the final dimensions of the Banach space using Green's type of imagery works [21.23] and presentations at conferences Ponta Delgada (Portugal) (2011) , Tbilisi (Georgia) (2011). Are found sufficient conditions for impulsive dynamical systems

$$\frac{dx}{dt} = A(t)x + f(t, x, y, z),$$
$$\frac{dy}{dt} = B(t)y + g(t, x, y, z),$$
$$\frac{dz}{dt} = C(t)z + h(t, x, y, z),$$
$$\Delta x|_{t=\tau_i} = x(\tau_i + 0) - x(\tau_i - 0) = E_i x(\tau_i - 0) + p_i(x(\tau_i - 0, y(\tau_i - 0), z(\tau_i - 0)),$$
$$\Delta y|_{t=\tau_i} = y(\tau_i + 0) - y(\tau_i - 0) = F_i y(\tau_i - 0) + q_i(x(\tau_i - 0, y(\tau_i - 0), z(\tau_i - 0)) ,$$
$$\Delta z|_{t=\tau_i} = z(\tau_i + 0) - z(\tau_i - 0) = G_i z(\tau_i - 0) + r_i(x(\tau_i - 0, y(\tau_i - 0), z(\tau_i - 0))$$

and partially linearized impulsive dynamic system

$$\frac{dx}{dt} = A(t)x$$
$$\frac{dy}{dt} = B(t)y + g(t, u(t, y), y, v(t, y,)),$$
$$\frac{dz}{dt} = C(t)z$$
$$\Delta x|_{t=\tau_i} = E_i x(\tau_i - 0) ,$$
$$\Delta y|_{t=\tau_i} = F_i y(\tau_i - 0) + q_i(u(\tau_i - 0, y(\tau_i - 0)), y(\tau_i - 0), v(\tau_i - 0, y(\tau_i - 0, y(\tau_i - 0)))$$
$$\Delta z|_{t=\tau_i} = z(\tau_i + 0) - z(\tau_i - 0) = G_i z(\tau_i - 0)$$

be locally and globally and dynamically equivalent. In this type of Difference Equations leading evolutionary type of partial differential equations, which are widely used in modelling of physical processes. To find sufficient conditions for impulsive dynamical systems dynamic equivalence is assumed that the linear part of the evolution operator satisfies the conditions and integrals separable members meet Lipchitz nonlinear terms to a small constant.

Works [3,15,34] and reports in Riga (2010) and Barcelona (2012) looks at Banach space without cropping Difference equations Autonomous

$$x(t+1) = g(x(t)) + G(x(t),\ p(t)),$$
$$p(t+1) = A(x(t))\,p(t) + \Phi(x(t),\ p(t)).$$

and supervised the event of such a system have invariant varieties neighbourhood. Noting that the two Difference equation systems are equivalent if there exists a homeomorphism (between the continuous and bijective representation) in a single system orbit Difference represents a Difference other systems in orbit and vice versa. The given approach allows a complex system of equations Difference qualitative research Difference replaced by simpler system of equations, often with smaller dimension of research. A variety of real problems in practice and leads to the reversible and the unedged Difference equation systems, the latter from a mathematical point of view is more complex and interesting at the same time. Were studied in the simplest unedged Difference equations, i. e such that the linear approximation is asymptotically stable. We found sufficient conditions, using Difference system right conditions for the given system there is a smooth variety Lipchitz invariant $p = u(x)$ - Centre for variety at a sufficiently small perturbations. Revealed a stationary point of existence theorems, the chosen functional space. Next on the basis of these results were found to be sufficient (close to the necessary conditions) for a given Difference equations equivalent to a simplified system.

$$x(t+1) = g(x(t)) + G(x(t), u(x(t))),$$
$$p(t+1) = A(x(t))\,p(t).$$

First, note that the resulting system's first Difference subsystem does not contain the variable p, while the second subsystem is linear to p. The proof is long enough and the theorem is based on the whole series. Successfully selecting the desired functional equations assistant, often importation of additional variables, as well as finding a suitable functional spaces was demonstrated adequate homeomorphism existence. The result obtained significantly generalizes over the world in the past mathematical literature results. The given paper is being prepared for publication deployed.

The results are summarized in "Conjugacy of discrete semidynamical systems in the neighbourhood of invariant manifold" [3], which was adopted for publication "Springer Proceedings in Mathematics."

2. Research on Rational Difference Equations

During the reporting period were investigated second order rational difference equations of the form:

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}}, \quad n = 0, 1, 2, \ldots \qquad (1)$$

where the parameters $\alpha, \beta, \gamma, A, B, C$ are nonnegative real numbers and the initial conditions x_{-1} and x_0 are arbitrary nonnegative real numbers such that $A + Bx_n + Cx_{n-1} > 0$ for all natural values of *n*.

The most important book on this kind of equations is *Dynamics of Second Order Rational Difference Equations with open problems and conjectures* (M.R.S. Kulenovic, G.Ladas, Chapman and Hall/CRC, Boca Raton, Fla, USA, 2002). It was used to study the known results about the behavior of solutions of the difference equation (1), for example, local and global stability, existence of periodic solutions, convergence of solutions, as well as to learn the main methods that are used in the quantitative research of the difference equations (methods of mathematical analysis – convergence of solutions, continuity; methods of mathematical logics – formulations of decisions and hypothesis; methods of theory of difference equations – different proofs for stability of solutions or statements that solution is unstable, ways of finding cycles and periodic solutions; numerical methods – hypothesis formulation based on numerical experiments with MS EXCEL and MATHCAD).

The before mentioned book contains many so called *Open problems* (problems that have not been solved) and conjectures, the main attention have been focused on the following open problem:

It is known that every positive solution of each of three equations

$$x_{n+1} = 1 + \frac{x_{n-1}}{x_n}, \quad n = 0, 1, \dots \tag{2}$$

$$x_{n+1} = \frac{1 + x_{n-1}}{1 + x_n}, \quad n = 0, 1, \dots \tag{3}$$

$$x_{n+1} = \frac{x_n + 2x_{n-1}}{1 + x_n}, \quad n = 0, 1, \dots \tag{4}$$

converges to a solution with (not necessarily prime) period-two: $\dots, \psi, \varphi, \psi, \varphi, \dots$.

In each case, determine ψ *and* φ *in terms of the initial conditions* x_{-1} *and* x_0 *. Conversely, if* $\dots, \psi, \varphi, \psi, \varphi, \dots$ *is a period-two solution for one of the equations (2), (3) or (4), determine all initial conditions* $(x_{-1}, x_0) \in (0; +\infty) \times (0; +\infty)$ *for which the solution* $\{x_n\}_{n=-1}^{\infty}$ *converges to this period-two solution.*

On the results of the research have been prepared reports and presented in conferences [17], [25], [28], [30], [33], also a publication [4] have been prepared and submitted.

During the reporting period have been collected and analyzed different publications about the latest results in the theory of difference equations.

Part of the seminars (direction “Technomathematics actual problems”) of the doctoral school “Research, modelling and mathematical methodology improvement for atomic and continuous media physical processes” of the University of Latvia have been devoted to the modeling and description of neural networks using difference equations. During the seminars have been analyzed a paper by Z.Zhou *Periodic orbits on discrete dynamical systems*, Computers and Mathematics with Applications, 45: 1155-1161, 2003, in which a single neuron model $x_{n+1} = \beta x_n - g(x_n), \ n = 0, 1, \dots$ with a simple signal function $g(x) = \begin{cases} 1, & x \ge 0 \\ -1, & x < 0 \end{cases}$ have been investigated. On a basis of this paper have been prepared and submitted a publication „Periodic Orbits of Single Neuron Models with Internal Decay Rate $0 < \beta \le 1$” (coauthors A. Aņisimova, I. Bula), where signal function is in the following form

$$g(x) = \begin{cases} -b, & x \le -\alpha \\ -a, & -\alpha < x < 0 \\ 0, & x = 0 \\ a, & 0 < x < \alpha \\ b, & x \ge \alpha \end{cases}, \ (b > a > 0 \text{ and } \alpha > 0).$$

3. Obtaining of a priori estimates for solutions of self-similar differential equations and their systems in order to ascertain the solvability of boundary value problems

The attempt to carry over difference or even though time scale equations the classical conditions which ensure a priori estimates of solutions and their derivatives of second order differential equations in explicit form for boundary value problems in accordance with aims of project was made in the report [14]. Mentioned classical conditions for the solutions of second order differential equations or their systems in explicit form are possible to formulate in the terms of so-called lower and upper functions. On the other hand a priori estimates for derivatives of solutions, having a priori estimates, ensure conditions of Schrader's or Nagumo's type or their generalizations. Generalizations of Nagumo's type conditions often are possible to express using one-sided estimates of differential equations right-hand sides. Conditions which guarantee a priori estimates for derivatives of solutions, having a priori estimates, in some situations similarly like using lower and upper functions are possible to express in the form of differential inequalities. Obtained a priori estimates of solutions and their derivatives of boundary value problem allow determining the solvability of boundary value problem, moreover various nonintersecting a priori estimates of solutions make possible to evaluate the number of boundary value solutions. It must be added that a priori estimates implies the stability of solutions, in addition the existence of different unstable solutions also are feasible.

Employment of the lower and upper functions in order to ascertain a priori estimates of solutions of difference and time scale equations are comparatively successfully approved in literature. It is not true with regard to conditions which ensure a priori estimates for analogues of solution, which has a priori estimate, derivative. The conditions formulated for this aim in the report [14] characterize certain formalism and nonconstructivity caused on this matter. There underlined exclusions on the behaviour of solutions for equation which is included in the boundary value problem. It must be stated taking into account the behaviour of right-hand site of equation, but this yet is unsolved task.

In parallel way with investigations in order to obtain conditions for a priori estimates of difference and time scale equations was extended the investigation of self-similar equations obtaining and solvability of their boundary value problems. These investigations were communicated in the reports [9], [13], [20], [37].

The example. The hydrodynamic flow in the boundary layer of rectangular channel is described by the system of partial differential equations

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0,$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2},\quad u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}$$

with the boundary conditions

$$u(x,0)=\beta\frac{\partial u(x,0)}{\partial y},\quad \frac{\partial \mathrm{T}(x,0)}{\partial \mathrm{y}}=-\frac{q_w}{k_\infty},$$
$$v(x,0)=0,\ u(x,\infty)=u_\infty,\ T(x,\infty)=T_\infty.$$

Using similarity transformation

$$f'(\eta)=\frac{u}{u_\infty},\quad \eta=y\sqrt{\frac{u_\infty}{\nu x}},$$
$$\Theta(\eta)=\frac{k_\infty(T-T_\infty)}{q_w x}\sqrt{\mathrm{Re}}$$

we obtain the boundary value problem for the self-similar differential equation system

$$\begin{cases} f'''+\dfrac{1}{2}ff''=0, \\ \Theta''+\dfrac{\mathrm{Pr}}{2}(\Theta' f-\Theta f')=0, \end{cases}\quad \begin{cases} f(0)=0,\ f'(0)=\kappa f''(0),\ \Theta'(0)=-1, \\ f'(\infty)=1,\ \Theta(\infty)=0, \end{cases}$$

$$\kappa=\frac{\beta}{x}\sqrt{\mathrm{Re}},\quad \mathrm{Pr}=\frac{\nu}{\alpha}.$$

Let us note, that the boundary conditions allow the flow sliding along wall of channel $(\beta\neq 0)$ and so one of the boundary conditions become nonclassical.

If we observe the previous boundary layer system of partial differential equations without the equation of temperature and introduce the flux function Ψ we obtain

$$u=\frac{\partial\Psi}{\partial y},\quad v=-\frac{\partial\Psi}{\partial x};$$

$$\frac{\partial\Psi}{\partial \mathrm{y}}\left(\frac{\partial^2\Psi}{\partial x\partial y}\right)-\frac{\partial\Psi}{\partial x}\left(\frac{\partial^2\Psi}{\partial y^2}\right)=\nu\left(\frac{\partial^3\Psi}{\partial y^3}\right).$$

Using self-similar variables we transform this equation in the form

$$\Psi(\mathrm{x},\mathrm{y})=\Psi_m(x)f(\eta),\ \eta=\frac{\mathrm{y}}{\delta(\mathrm{x})};$$

$$f'''=\frac{\delta\Psi_m}{\nu}(f'^2-ff'')-\frac{\Psi_m\delta'}{\nu}f'^2.$$

In order to obtain the self-similar ordinary differential equation must be fulfilled the following expressions

$$\frac{\delta\Psi_m}{\nu} = a = const \ , \quad \frac{\Psi_m \delta'}{\nu} = b = const \ ,$$

So, after differentiation this self-similar equation obtains the following view

$$f^{(IV)} + aff''' + (2b - a) f'f'' = 0.$$

Considering intrinsic heat convection along thin heated plate which is embedded in the porous medium one obtains the boundary value problem of de la Vallee-Poussin type for the self-similar equation (Z.Belhachmi et al, On the Family of Differential Equations for Boundary Layer Approximations in Porous Media. European Journal of Applied Mathematics, 12, 2001, 513-528)

$$f^{(iv)} + \frac{1+\beta}{2} ff''' + \frac{1-3\beta}{2} f'f'' = 0;$$

$$f(0) = 0, \quad f'(0) = 1; \quad f(\infty) = 1; \quad f'(\infty) = 0.$$

Physical interest cause solutions of this problem which satisfies the estimates

$$0 \le f'(t) \le 1, \quad t \in [0, +\infty).$$

In general we can consider the differential equation

$$f^{(iv)} + g_1(\beta) ff''' + g_2(\beta) f'f'' = 0,$$

with continuous coefficients. Considered boundary value problem for this equation has exactly one solution with physical meaning, if

$$\frac{1}{2} \le g_1(\beta) < +\infty; \quad -\infty < g_2(\beta) \le \frac{1}{2},$$

on the other hand there are not solutions of this problem in the case

$$-\infty \le g_1(\beta) \le \frac{1}{4}; \quad \frac{5}{4} \le g_2(\beta) \le +\infty.$$

Remaining are two cases

$$\frac{1}{4} < g_1(\beta) < \frac{1}{3}; \quad 1 < g_2(\beta) < \frac{5}{4};$$

$$\frac{1}{3} \le g_1(\beta) < \frac{1}{2}; \quad \frac{1}{2} < g_2(\beta) \le 1.$$

At the first of cases the boundary value problem has not solution with physical meaning, but in the second of these cases the boundary value problem has at least one solution with physical meaning.

4. Hyperbolic Heat Conduction Equation and its Applications

S. Blomkalna began to work in the project as a participant in leading researcher's A. Buikis group. She started PhD studies in October, 2011 with J. Cepītis as a thesis adviser and A. Buikis as a consultant. Significant part of her research, cooperating with A. Buikis, was multiple model development for intensive steel quenching process and analytical-numerical methods for their solutions. Particular attention was paid to inverse problems, since it is experimentally impossible to determine initial heat flux. Hyperbolic Heat Conduction equation (HHCE) has promising properties for better mathematical description of metallurgical processes. This work is a part of a series of studies devoted to Intensive Quenching models. Results of obtained results are presented in conference thesis [29, 35] and paper [6].

The second goal was to find, identify, analyse and examine other applications of HHCE.

The review identifies material heating, using ultra short impulse lasers as one of the most popular fields where HHCE is used and intensively studied. There are several studies about processes in thin gold film. Various impulse functions, different sample shapes and multilayered objects are investigated. Mathematical models include classic (parabolic) heat conduction equation, hyperbolic equation and dual phase lag equation in one, two and three dimensions. Several methods of solving problems are proposed and analysed; stability analysis is done for 1D case. HHCE can be successfully applied to describe laser heating of biological materials - human tissue in liver, skin, kidney etc. in surgical procedures, usually with short, localized heat source. There are reports where HHCE also suits for forest fire modelling, virus infection spread and pollution diffusion in harbours.

5. Electron flow modelling gyrotrons.

Researchers gathered a group of recognized experience in nonlinear Schrödinger-type equations, qualitative and numerical solution to simulate electron motion gyrotrons required studies and calculations of thermo-nuclear reaction control a new type of nuclear reactors.

The mathematical model was reduced to nonlinear complex Schrödinger-type partial differential equations system that describes one or more electrons RF field amplitude $f(x,t)$ oscillations in gyrotrons and transversal orbital momentum $p(x,t)$ depending on the time t and from x at the segment $[0, L]$.

High-frequency RF field amplitude $f(x,t)$ in transversal resonator and the electron orbital momentum $p(x,t)$ with parameters ϑ $(0 < \vartheta \leq 2\pi)$ can be described by a complex system of differential equations

$$\frac{\partial p}{\partial x} + i(\Delta + |p|^2 - 1 - g_b(x))\, p = if(t,x)$$

$$\frac{\partial^2 f}{\partial x^2} - i(1+\delta_\omega)\frac{\partial f}{\partial t} + (1 + 0{,}5(\delta_\omega + g_c(x)))\, g_d(x)\, f = I(1+\delta_\omega)(1+g_c(x))^2 < p > .$$

where i is the imaginary unit, $0 \leq x \leq L$ and $0 \leq t \leq t_f$ are the normalized axial and temporal coordinates, L is the exit from the interactive space, t_f is the final time, Δ is the cyclotron resonance mismatch, δ_ω is a frequency mismatch, I is the dimensionless beam current parameter, $g_b(x)$, $g_c(x)$, $g_d(x)$ are given empirical functions, $< p >$ is averaged value of p.

The system has to be supplemented by the initial condition $p(t,0,\theta_0) = \exp(i\theta_0)$ with the parameter θ_0, $f(x,0) = f_0(x)$, and with boundary conditions at the entrance and exit of the interaction space $f(t,0) = 0$, $\frac{\partial f(L,t)}{\partial x} = -i\gamma f(L,t)$, where $f_0(x)$ is given a complex function and γ a positive parameter. The

$$\eta = 1 - \frac{1}{2\pi}\int_0^{2\pi} |\, p(t,L,\theta_0)\,|^2 \, d\theta_0$$

is the electron perpendicular efficiency which describes the transfer of the electron orbital momentum from the beam of RF.

The equations takes into account the dependence of the electron relativistic factor on the axial coordinate and dependence of the magnetic field on the axial coordinate: electron relative factors in relation to the axial axis and the dependence of the magnetic field. Usually these dependencies are weak and are ignored. The difficulties arise in solving the nonstationary problem for large time interval with the oscillating complex initial function.

We found the law of conservation of power

$$I\eta = \frac{dW}{dt} = 2\delta_1\gamma\,|\, f(t,L)\,|^2 - 2I\,\mathrm{Im}\int_0^L (1 - g_2(x))\, f^* < p > dx\,.$$

Gyrotron equation solution structure is very complicated. To different parameter values significantly changes the picture of the phase space picture. We find the topology of different kinds of oscillations of a gyrotron in the $\Delta - I$ plane (see Figure 1). White regions correspond to stationary oscillations, gray regions correspond to automodulation, and dark regions to chaotic oscillations. The contours of constant efficiency are shown by the dashed curves. The point of the maximum efficiency is marked by the cross.

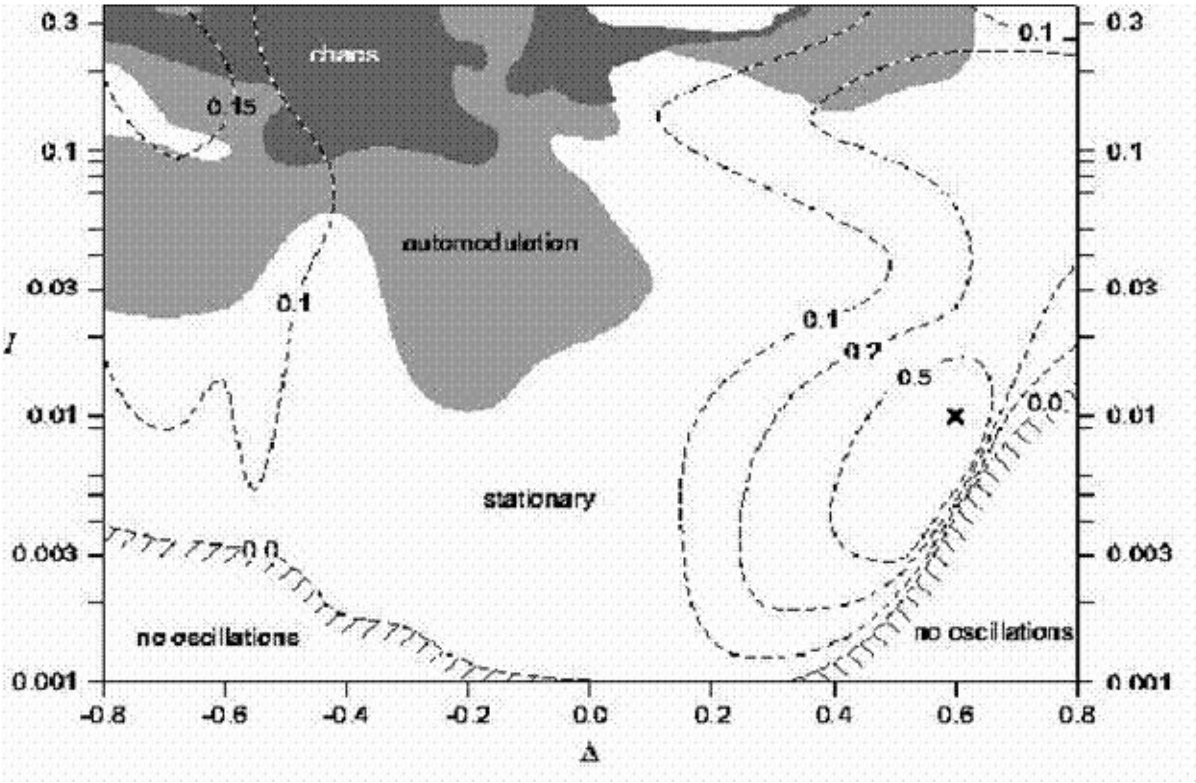

Figure 1

It turned out that the approximation space with central differences (DS-2) and the implicit of difference scheme is unstable for modified version of gyrotron equation, because the solution by reducing the time step was oscillating in time as well as space. To find the solution with the physics method was used straight lines, reducing the partial differential equations of the system of ordinary differential equations and solving them with the MATLAB solver, where the time step is selected automatically according to the given precision. Numerical calculations showed that the stationary solution f to fluctuations in the space is really noticeable, while the time is not (see Figure 2).

In collaboration with physicists in the analysis of single-mode equations for gyrotrons, found that at small time steps oscillations with increasing amplitude of the electron source is relativistic factor and dependence on the magnetic field (see Figure 2).

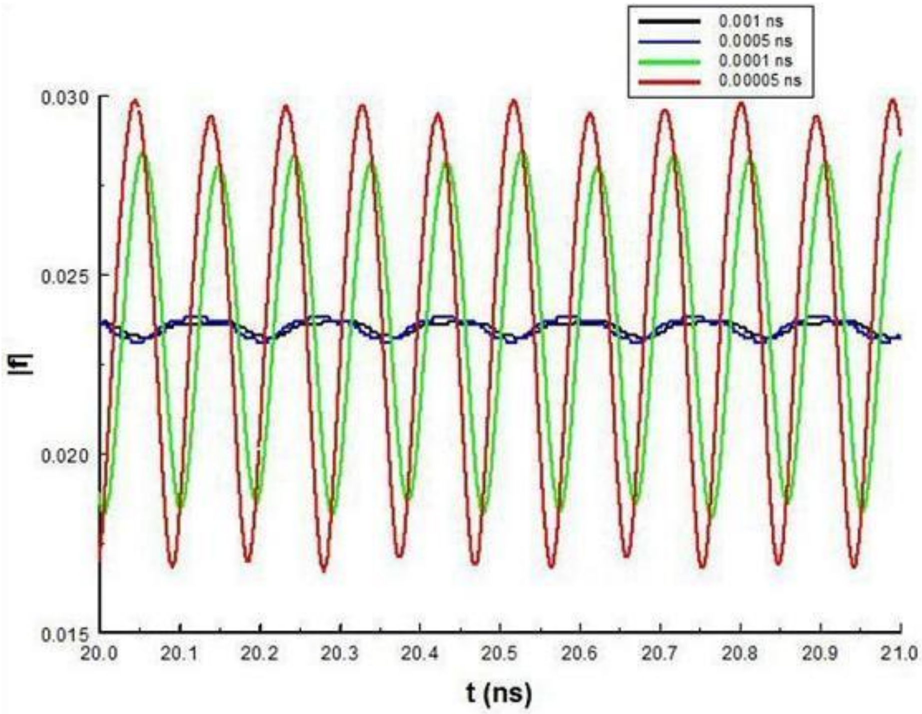

Figure 2

The results are given in the papers [1, 2] at SCI Expanded journals "Mathematical Modelling and Analysis" and "Nonlinear Analysis: Modelling and Control" and presented [18] at the conference in Sigulda (2011).

6. Nonlinear plasma flow perturbations

Larger scale plasma instabilities not leading to an immediate termination of a discharge often result in periodic nonlinear perturbations of the plasma. A minimal possible physical model has been formulated for description of the system with drive and relaxation processes which have very different time scales. The model is based on two equations: the first being responsible for the relaxation dynamics (MHD force balance) and the second for the drive (energy conservation).

$$\begin{cases} \frac{d^2}{dt_n^2}\xi_n = \left(p_n' - 1\right)\cdot\xi_n - \delta\cdot\frac{d}{dt_n}\xi_n \\ \frac{d}{dt_n}p_n' = \eta\cdot\left(h - p_n' - \alpha\cdot\xi_n^2\cdot p_n'\right) \end{cases} \quad (1)$$

The first equation is the equation of instability dynamics. It describes fast events occurring in a system due to instability growth. It is natural to assume that the amplitude of the displacement of the magnetic field ξ corresponding to this instability is the main characteristic variable of the equation. Here p is the plasma pressure gradient and δ is dissipation.

The second equation is the equation for the pressure gradient. It describes the power balance. The h represents the normalized power input into the system and is responsible for inducing the burst. The η is the characteristic relation between the two heat diffusion coefficients

For a convenient mathematical analysis this model can be represented as an autonomous system of three parameter-dependent first order ordinary differential autonomous non-linear equations

$$\begin{cases} \dfrac{dx}{dt} = (z-1)\,y - \delta \cdot x \\ \dfrac{dy}{dt} = x \\ \dfrac{dz}{dt} = \eta \cdot \left(h - z - y^2 z\right) \end{cases} \qquad (2)$$

Differential system (2) contains the square and cubic nonlinearity. For the first time differential equations (1) published the Max Planck Institute for Plasma Physics researchers D.Constantinescu, O.Dumbrajs, V.Igochine, K. Lackner, R.Meyer-Spasche, H.Zohm in the paper *A low-dimensional model system for quasi-Periodic plasma perturbations*, Physics of Plasma 18 (2011), No.6.

System obtained is somewhat similar to the Lorentz equation system, which has a very complicated phase picture. Firstly, the system is dissipative, i.e. $div\,(X) = -\delta - \eta - \eta y^2$. This means that the flow generated by the system of equations compressed volumes. But this in turn means that there are no invariant torus of trajectory formed by a quasi-periodic solution. Other hand, there are attractors and trajectories asymptotically approached them. The second characteristic is that the system is symmetric with respect to the z axis.

Differential equations system (2) has three fixed points $P_1 = (0,0,h)$, $P_2 = (0,\sqrt{h-1},1)$ and. $P_2 = (0,-\sqrt{h-1},1)$. Stationary point P_1 is asymptotically stable, if $0 < h < 1$. If $h > 1$ then the point P_1 is a saddle point with a two-dimensional stable and one-dimensional unstable manifold. While the symmetrical stationary points P_2 and P_3 are asymptotically stable if $\delta h(\delta + \eta h) > 2(h-1)$ and unstable if $\delta h(\delta + \eta h) < 2(h-1)$ with one-dimensional stable and unstable two-dimensional unstable manifold.

To prove this result we find Jacobi matrix of stationary points, find and characteristic polynomials and using the Routh-Hurwitz criterion determine the sign of the real part of the eigenvalues.

Numerical experiments show that the system of differential equations have different type of bifurcation. Having used the analytical method other hand, may find that the pitchfork bifurcation in stationary point P_1 , and the Andronov-Hopf bifurcation in points P_2 and P_3. Specifically, using the centre manifold theory with the reduction principles can be stated that the stationary point P_1 is a locally asymptotically stable if $h = 1$ and supercritical pitchfork bifurcation point. While the stationary points P_2 and P_3 are supercritical or subcritical Andronov-Hopf bifurcation points according to the first Ļyapunov coefficient signs. This fixed point is surrounded by small stable or unstable limit cycles, corresponding to periodic solutions to them. If $h > 1$ then the stationary point is globally asymptotically stable.

Numerical experiments show that the system of differential equations (2) with the parameters $\delta = 0.845$, $\eta = 0.0845$, $h = 1.7$ and initial values $(x_0, y_0, z_0) = (0.1, 4.1, 1.4)$ have a strange attractors. See Figure 3.

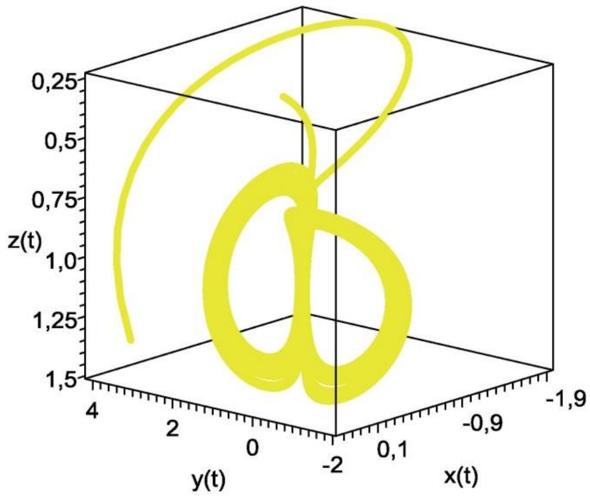

Figure 3

The numerical results allow distinguish five dynamical zones of oscillations:

1. damped oscillations
2. simple periodic oscillations
3. double periodic oscillations
4. existence of chaotic oscillations
5. sawtooth type oscillations with a long rise time and a short crash time. See Figure 4.

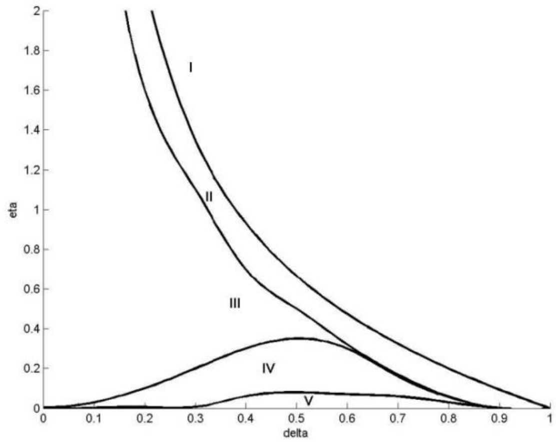

Figure-4

We studied the phase portrait of given system for different parameter values. The results are given in the works [26, 31, 38] and presented at conferences in Jelgava (2012), Tallinn (2012) and Novacellā (Italy) (2012). In addition, there was developed thesis (S.Avdejevs) (2011).

7. Research on fiber suspension rheology

Another part of the project was devoted to research on fiber suspension rheology. Contributions have been made to mathematical modeling, qualitative analysis and application-oriented numerics for short fiber suspension flows. The principal directions of our work include modeling the wall effect on fiber orientation; analysis of stability of pressure driven fiber suspension flow in a domain between parallel plates; the proper generalized decomposition (PGD) method as an universal technique for solving PDEs in high dimensional domains, with the particular application to the kinetic equations appearing in rheology.

Modeling the wall effect. It is well known that fiber dynamics near the wall are different from the dynamics in the bulk of suspension due to various effects that include long and short range interactions mediated by the fluid phase as well as direct mechanical contacts between the fiber and the wall. The current state-of-art models used in simulations of industrial processes either ignore the wall effect completely or employ quasi 2D simulations of fiber orientation in regions near the wall. The high number density of fibers in typical applications in, e.g., fiber-reinforced thermoplastics effectively prohibit executing direct microscale simulations, thus averaged mesoscale models are called for. We have developed novel microscale and mesoscale models based on simplifying assumptions on the nature of fiber-wall interactions.

The dynamics of a suspended fiber in microscale can be described by the Jeffery's equation. In close proximity to a wall Jeffery's model may predict geometrically impossible orientation states in which the fiber would penetrate the rigid wall. We have derived a generalization of Jeffery's equation that describes the time evolution of the unit vector p representing the orientation of the central axis of the fiber in the form

$$\dot{p} = \begin{cases} \dot{p}_J := (I - p \otimes p) \cdot Mp, & p \cdot n > d/l, \\ \dot{p}_J - (n \cdot p_J) \frac{n-(p\cdot n)p}{\|n-(p\cdot n)p\|^2}, & p \cdot n = d/l. \end{cases}$$

Next to the orientation vector p this model contains another parameter d, which measures the distance of the centre of mass of the fiber to the wall. This model presumes that the fiber evolves according to Jeffery's equation in the absence of contact and slides along the wall in case of contact. Starting from this equation, a range of more complicated models has been derived.

The effects of lubrication prevent direct mechanical contacts between the fiber and the wall at low concentration regime. In a nutshell, the lubrication effect goes as follows: as a part of the fiber approaches the wall, some mass of fluid must be displaced, thus inducing an effective repulsive force acting on that part of the fiber depending on the relative velocity. The contact condition is replaced by additional term in Jeffery's equation.

The fiber concentration in technical suspensions is high enough to make fiber-fiber interactions count. The classical Folgar-Tucker and related models account of this complicated process by introducing orientational diffusion. In the microscale this leads to stochastic ODE – a Langevine equation. Additionally to the orientational diffusion we add a diffusion of the fiber-wall distance parameter due to random inter-fiber interactions in our model.

Moreover, the structure of the generalized Jeffery's equation allows to incorporate other effects, such as mobility coefficients depending on the fiber shape.
Extensive numerical simulations have been performed for local flow fields of significance for practical applications using different variations of the extended microscale model. The solution complies with the geometrical restrictions in near-wall regions (which are typically violated using pure Folgar-Tucker model) and give good qualitative agreement with the published orientation states observed experimentally. The group of Soderberg has published experimental results of fiber orientation for different surface roughness, a parameter that strongly affects the fiber-wall collisions. Further work on our model is needed to improve the quantitative agreement with these experimental results.

Mesoscale (kinetic scale) models are derived with respect to a distribution function defined on the Cartesian product of the time domain, space domain and the phase space of orientational and other additional state variables. For instance, the Folgar-Tucker model is the Fokker-Planck equation derived from Jeffery's equation with the assumption of isotropic orientational diffusion with intensity proportional to the local macroscopic shear rate. Two strategies to incorporate wall effect in mesoscale models are as follows: deriving a new Fokker-Planck equation from a suitable microscale model, or to modify the Folgar-Tucker equation phenomenologically so that to avoid non physical orientation states near the wall. Since the Folgar-Tucker and related models are expressed in terms of low-order moments of the distribution function instead of the function itself, the exact state of orientation of single fibers is not known, and nor is the exact moment of fiber-wall contact.

To reach that end, one checks whether the orientation state predicted by a classical model (not including wall effects) violates the geometrical restrictions. If so, a projection step is made to phenomenologically represent the sliding of fibers along the wall in case of contact. The direction of the projection is determined by interpolating between two directions, the first being exact if the exact distribution function is a linear combination of Dirac delta-distributions with support on the directions of eigenvectors of the orientation tensors, and the other being exact if the exact distribution is identical to its up to second order expansion in spherical harmonics. The interpolating coefficients depend on the determinant of the orientation tensor as a measure of planarity of the distribution in a similar way as the hybrid closure approximation is constructed. The model has been tested by implementation in the framework of CoRheoS solver developed at Fraunhofer ITWM, Kaiserslautern, Germany. Good agreement with experimental results is achieved.

The generalized Jeffery equation leads to a completely different kind of Fokker-Planck equation than in the Folgar-Tucker case, namely, it is a PDE on a manifold with an edge parametrized by the states of fibers that are in contact with the wall. Depending on the collision model, the solutions can be true distributions, i.e., the edge (a set of zero measure) carry positive "mass" of orientation distribution.

Various numerical techniques for solving advection-diffusion equations on (submanifolds of) sphere have been studied. For instance, solutions of linear equations can be sought as series of spherical harmonics. The coefficients determining the advection velocity being polynomial, the elliptic operator on the left-hand side of the PDE maps the space of finite linear combinations of spherical harmonics to itself, and the coefficients can be found by recursive application of certain transformation rules. This not only leads to efficient numerical methods, but also provides direct means to estimate the error of moment expansion of the Fokker-Planck equation and application of a certain closure model.

Stability of channel flows. Linear stability analysis of the channel flow of a fiber suspension in a channel domain using the Folgar-Tucker model has been performed by Lin Jianzhong et. al. We have extended the analysis to the FTMS model, a generalization of Folgar-Tucker model for highly concentrated suspensions. The orientation of fiber orientation tensor under the influence of a given external velocity field is given by the equations

$$\begin{aligned}\frac{D}{Dt}a^{(2)} &= a^{(2)}\cdot M + M^\top \cdot a^{(2)} - (M+M^\top) : a^{(4)} \\ &\quad + \dot{\gamma}\left\{C_i(I-3a^{(2)}) + S_0(a^{(2)}\cdot a^{(2)} - a^{(2)} : a^{(4)})\right\}\end{aligned}$$

We use the standard notation for Folgar-Tucker model. The matrix M is a linear combination of the global velocity gradient and its transpose, and the velocity is governed by the incompressible generalized Navier-Stokes

$$\frac{\partial u}{\partial t} + u\cdot\nabla u = -\rho^{-1}\nabla p + \nabla\cdot(\tau_s + \tau_f)$$

$$\tau_s + \tau_f = Re^{-1}(\nabla u + \nabla^\top u) + N_p \nabla u : a^{(4)}.$$

Pressure driven channel flow with isotropic fiber orientation state at the inlet exhibits slowly decaying transient effects connected with the fishbone pattern of fiber orientation, i.e., in certain parts of the channel the preferred orientation of the fibers is oblique to the global velocity field, thus strongly increasing the local value of effective viscosity. However, further downstream a stationary state is reached, corresponding to parabolic velocity profile (Poiseulle flow). Small perturbations of the stationary flow induce certain transient effects. The time behaviour of these transients is the object of study of stability analysis.

By decomposing the perturbation in Fourier components and linearizing the equations one obtains a boundary value problem for a fourth-order ODE with respect to the stream function of each component of the perturbation depending on the wave-number and direction of travel of the harmonic perturbation. The rate of decay or growth of the component enters the equation as a complex generalized eigenvalue. The result of the analysis may be interpreted as a generalization of Orr-Sommerfeld problem: find complex values of the parameter c, for which the boundary value problem with homogeneous Dirichlet and Neumann boundary conditions for

$$\sum_{k=0}^{4} J_k(y, \alpha, Re, C_i, S_0, c)\partial_y^k \phi(y) = 0$$

admits a non-trivial solution. The stability of the flow depends on the sign of the imaginary part of the generalized eigenvalues c.

The problem is non-linear with respect to c and must be solved for multiple sets of parameters, so efficient numerical methods are required. Several numerical strategies have been tested. The best results were obtained by either using the built-in solver 'bvp4c' or implementing a finite difference method for the linearized equation in a Chebysheff grid.

Curves of neutral stability of the flow have been computed for various sets of physically relevant values of the parameters have been obtained showing the stability regions in the plane of wave number of the perturbation vs Reynolds number of the flow. Our results confirm the experimental data in demonstrating that the presence of fibers increases the stability region of the channel flow.

Applications of PGD method. PGD (proper generalized decomposition) method for numerically solving multi-dimensional PDEs on tensor product geometries is rapidly growing in popularity in the recent years. In sharp contrast to conventional methods (such as finite element or finite volume) where the number of degrees of freedom increases exponentionally with the dimension of space, the PGD method often exhibits only a linear growth of number of degrees of freedom with dimension of the space. Among the applications of this method is the Fokker-Planck equations in rheology. Depending on the details of modelling, these equations can be formulated on phase spaces of arbitrary high dimension, where exceeding 100 dimensions is by no means an exception. The PGD method is the only feasible way to solve such equations in terms of computational costs and memory requirements.

The numerical solution is sought as the sum of products of single argument functions:

$$u(x_1, \ldots, x_n) \approx \sum_{k=1}^{m} \prod_{i=1}^{n} U_i^k(x_i),$$

where the basis functions are computed recursively. Plugging the ansatz into the equation, multiplying by a test function and integrating over all coordinates but one, a weak form for an equation with respect to a basis function is obtained. It should be noted that the multiple integral can be computed as the product of 1D integrals for functions admitting this kind of representation. The n-D PDE is solved by solving multiple nonlinear ODEs. Another advantage of the PGD method is the ability to cover a range of values of multiple parameters.

The first step to apply PGD method for the equations of suspension rheology has been studies of simpler equations such as Laplace, heat transfer and wave equations. Several strategies of implementation have been tried out. The influence of the choice of initial guess on the rate of convergence of the inner iterations has been studied for different kinds of equations and other data (such as source functions etc).

The equations of rheology are solved on tensor product geometries involving unit spheres. The parametrization of the sphere using spherical coordinates transforms maps the sphere to a rectangular domain allowing a direct application of the PGD method. Experimental versions of codes for application of PGD method to PDEs on tensor product of spheres have been developed. The work on this part of project will be continued.

Other topics. The qualitative properties of equations of fiber suspension flows such as existence, uniqueness and stability in Bochner spaces, have been studied using methods of functional analysis. By improving the formulation of an inequality and simplifying the proof, a theorem of existence and uniqueness of a weak solution of the system of mesoscale rheological equations has been slightly generalized. The demands on the source functions have been relaxed from a polynomial growth to a condition on the smoothness.

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