EFFICIENT NUMERICAL METHODS FOR DIFFERENTIAL PROBLEMS WITH VARIABLE LAYER COEFFICIENTS ON DERIVATIVES

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Aim of the research sub-activity: Development of specific methods of numerical analysis and applications in sophisticated problems of mathematical physics and development of improved, new, and advanced numerical methods for engineering problems.

General: implementation of study's purpose

When modelling applied mathematical physics, hydrodynamics and magnetohydrodynamics problems, we need to numerically solve partial differential equations and their systems with in layers different coefficients on derivatives. The presence of these parameters, as well as the complex geometry of the solution area creates additional application difficulties for universal well-developed numerical methods and software packages and they become ineffective. Therefore, advanced special numerical methods must be developed, with good qualities of precision and easy on implementation in addressing the problems allowing parameters from wide range of changes.

In the project research was initiated and continued into new special difference schemes for numerical modelling of ordinary and partial differential equations, which are based on the exact (differential) spectrum (eigenvalues and eigenfunctions) for the application on spatial 2nd order derivative substitution with finite differences, creating differential schemes with exact spectrum (DSES). The particular matrix of finite differences is reduced to the canonical form by using spectrum of final differences. Canonical form diagonal matrix of order M discrete eigenvalues are replaced by the first M eigenvalues of differential operator, forming DSES matrix. Algorithms based on the method of lines, leaving derivatives for other directions continuous, were implemented for different 2D problems of mathematical physics (linear and non-linear). Partial differential equation problems with different condition types on borders were studied. The developed algorithms are implemented with the help of MATLAB software, solving linear and nonlinear problems in differential equations.

Problems of mathematical physics with periodic boundary conditions were researched specifically. Then circulant matrix algorithms can be created, which allow to simplify the calculations. DSES are particularly effective for Dirichlet and periodic boundary conditions, because in these cases M discrete eigenvectors matches with grid values of the first M eigenfunctions of the differential problem.

For the first time DSES idea is mentioned in the literature in 1975 by Ukrainian scientists V. Makarov and I. Gavrilyuk in work " On constructing the difference net circuits with the exact spectrum" Dopov. Acad. Nauk Ukr. RSR, Ser. A p. 1077-1080, 1975 (in Ukrainian), looking at a linear heat conduction problems with homogeneous Dirichlet boundary conditions. It was possible to transfer this method to various problems of mathematical physics with different boundary conditions (Dirichlet, Robin and periodic) [1,4,5,6,7,8,23,25,28,31,33]. Efficiency of the method was compared with the standard differential approximation schemes of the 2nd order

(DS-2) and higher order difference approximation schemes (DS-p, p = 4, 6, 8) for problems with periodic condition.

Especially efficient it was for DSES problems with periodic boundary conditions. In the duration of the project doctoral dissertation "Numerical modelling of mathematical physics problems with periodic boundary conditions" of LU student Aigars Gedroics was designed, which will be defended in 2013. His research results, together with co-authors have been published in five papers [2,3,7,16,30] (article [33] accepted for publication), and 10 theses [1,6,13,14,24,23,25,27, 28]. Various ordinary and partial differential equation problems with Dirichlet boundary conditions using DSES have been researched by Master of Faculty of Physics and Mathematics Sergejs Rogovs. As a result a publication with co-author [15] has been published (article [33] accepted for publication) and 3 theses [23,25,28]. The second Master Maksims Marinaki by solving the problem of fluid flowing around periodically positioned cylinder in magnetic field is co-author of the article [30], paper [32] submitted for publication, as well as a theses [27]. DSES ideas and examples of usage will be included in the teaching material for Masers of the Faculty of Physics and Mathematics [34].

Another direction of research is related to the development of numerical methods for engineering-type calculations, by reducing the complex mathematical models to simpler. The development of these algorithms is based on different mathematical method applications and proves. The developed numerical methods are applied in sub-activity No. 4.4.1. “Unconventional use of classical mathematical physics methods in mathematical modelling" for various engineering problems. In the project today's technology-specific processes were researched and modelled. From the practice in these models often appear large or small geometrical or physical parameters, jumps of differential equation coefficients, etc. The direct and inverse problems of mathematical physics nowadays must be practically and theoretically analyzed, taking into account their specificities. Had to develop special methods and approaches suitable for specific class of problems, because the general analytical and numerical methods are not always sufficiently effective and accurate. For analysis and modification of the formulation and exact and approximated solution construction for problems of mathematical physics, exact and approximated (including final differences) analytical and numerical methods were used. This includes modelling of applied problems which are important in the practice: intensive steel quenching technology – theoretical models, analysis and exact and approximated solutions [5,8,17]; modelling of new type energy devices based on the principles of vortex effect [2]; modelling of the metal particles in peat layers [14,16,24]; motion modelling of extended magnetized droplet and thread in rotating magnetic field [9,10,11,18,19,26,29]; nonlinear heat flow modelling [7]; the MHD fluid flow models – creation, analysis, calculations [3,22,27,30,32], the electron flow model in gyrotron – creation, analysis, and numerical simulations [12,20,21].

In the engineering-type calculations together with special difference schemes the conservative averaging method of prof. Andris Buikis were also used. This can reduce the dimension of the original problem [14,16,24,33].

The research results are presented in 13 publications [2,3,7,8,9,15,16,17,18,19, 20,21,30], one article is accepted for publication [33], 3 have been submitted for publication [29, 31.32]. Totally 16 papers presented: 9 papers in international MMA (Mathematical Modeling and Analysis) Conference - Druskininkai (Lithuania) 2010 May 26-29 [5], Sigulda (Latvia) 2011 May 25-28 [10,11,12,13,14], Tallinn (Estonia) 2012 June 6- 9 [25,26,27], 1 paper in 6th International NAA'12 (Numerical Analysis and Applications) conference Lozenetz (Bulgaria) 2012 June 15-20 [28], 1 paper in 16th International ICDA2010 (Difference Equations and Applications) conference in Riga in 2010 June 19-23 [6], 2 reports in the 8^{th} Latvian Mathematics Conference 2010 Valmiera April 9-10 [1,4] 3 reports in the 9^{th} Jelgava Latvian Mathematics Conference 2012 March 30-31 [22,23,24] (see theses).

When creating a new type of algorithms for problems with different coefficients on the derivatives H. Kalis abstract "The Finite difference schemes with exact solution spectrum for some heat transfer problems" was prepared for the 5th international NAA'10 conference "Finite-Difference Methods: Theory and Applications", Lozenetz (Bulgaria) 2010 June 28—July 2. The author, however, did not participate in the conference as abstracts did not reach the recipient.

Detailed outline of study

Further basing on the published papers and theses results of the studies will be described and analyzed for new and improved algorithms and concrete applied problems in mathematical modelling.

1. Difference schemes with exact spectrum (DSES) [1,4,6,13,15,23,25,28,31,33].

We will look at three types of boundary conditions while developing DSES as an effective method for solving ordinary and partial differential problems: periodic, Dirichlet, Robin and homogeneous boundary conditions.

1.1. Problems with periodic boundary conditions [13,33]

When building DSES with periodic boundary conditions it is important to use the advantages of using a circulant matrix.
It is easy to create a MATLAB algorithms with circulant matrices – finding inverse matrix, matrix multiplication, and matrix multiplication by a vector.
Circulant matrix with dimensions *nxn* in matrix form

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-2} & a_{n-1} & a_n \\ a_n & a_1 & a_2 & \cdots & a_{n-3} & a_{n-2} & a_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_3 & a_4 & a_5 & \cdots & a_n & a_1 & a_2 \\ a_2 & a_3 & a_4 & \cdots & a_{n-1} & a_n & a_1 \end{pmatrix}$$

It can be expressed with its first row:
$A = [a_1;\ a_2;\ a_3; \dots;\ a_{n-1};\ a_n]$.
For these matrices simplified algorithms for matrix operations are possible that can be performed in a relatively small computation time – add, multiply, finding the inverse matrix. Also finding eigenvalues and eigenvectors of matrix is simplified.
Sum and product of such matrices produces a circulant matrix. Also, if the inverse matrix for the circulant matrix exists, it is also circulant.
MATLAB programs with definition of circulant matrix and their operations – addition, multiplication and calculating the inverse matrix – was created by doctoral student at the University of Latvia Aigars Gedroics.

The eigenvalues of matrix A can be expressed in form $f(\omega^k)$, $k = \overline{0, n-1}$, where ω is primitive *n*th root of unity – solution $e^{2\pi i/n}$ of equation $x^n - 1 = 0$,

$$f(\lambda)=\sum_{i=1}^{n} a_i\lambda^{i-1}$$

Eigenvectors of the matrix are in form $(1,\omega^k,\omega^{2k},\ldots,\omega^{(n-1)k})^T$, $k=\overline{0,n-1}$.
Circulant matrices has also a property
$A=WDW^*=W^*DW$,
where W is matrix which consists of matrix A eigenvectors in columns of the matrix, matrix D – diagonal matrix with eigenvalues on the diagonal. Matrix W^* is complex conjugate matrix of W.
As an example we can look at Poisson‘s 1D equation problem with periodic boundary conditions:
$-u''(x)=f(x), u(0)=u(L), u'(0)=u'(L)$.
This problem always has a unique solution for any Riemann integrable function $f(x)$, if the following condition is fulfilled

$$\int_0^L f(x)dx=0$$

and the value of the unknown function $u(x)$ is fixed in some point:
$u(x_0)=u_0,\ x_0\in[0,L]$.
It is possible to express solution of the problem analytically.
In the beginning it is possible to find a solution for a problem when $x_0=0$ and $u_0=0$. This is in form

$$\bar{u}(x)=\int_0^x (t-x)f(t)\,dt-\frac{x}{L}\int_0^L t\,f(t)\,dt$$

In the result of simple transformation we can find the solution also for the general problem

$$u(x)=\bar{u}(x)-\bar{u}(x_0)+u_0$$

We will use the 3-point stencil for the problem by approximating the second order derivatives with central differential:

$$-u''(x_j)\approx\frac{-u(x_{j-1})+2u(x_j)-u(x_{j+1})}{h^2}$$

The original problem with N algebraic equations (DS-2), can be rewritten in the form
$Ay=f$,
where A is matrix with dimensions NxN

$$A=\frac{1}{h^2}\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & -1\\ -1 & 2 & -1 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & -1 & 2 & -1\\ -1 & 0 & 0 & \cdots & 0 & -1 & 2\end{pmatrix}$$

where y, f are column-vectors of order N, $y_j\approx u(x_j)$ $f_j=f(x_j)$.
The matrix A is circulant, i.e. it can be defined by its first row and the rest of the rows are copies of the first row shifted by one element. Therefore, it can be described in form:

$$A=\frac{1}{h^2}[2;\ -1;0;\ldots;0;0;-1]$$

From the spectral problem of matrix A it follows that the eigenvalues are

$$\mu_k=\frac{4}{h^2}\sin^2\frac{k\pi}{N}$$

and w^k can be expressed in form

$$w^k{}_j = \sqrt{\frac{1}{N}}\exp\left(\frac{2\pi ikj}{N}\right)$$

Scalar product of vector w^k and complex conjugate vector $w^k{}_*$ equals with Kronecker delta $\delta_{k,m}$.

From property of circulant matrices, matrix A can be expressed in the form $A = WDW^*$.

By using this expression, problem $Ay = f$ can be transformed in the form $Dv = W^* f$, where $y = Wv$.

For the indices $j = \overline{1, N-1}$ values of the new problem can be solved for unknown argument v:

$$v_j = \frac{1}{\mu_j}(W^* f)_j$$

For $j = N$ we get an expression which is consistent with the condition of problem's solution existence $\int_0^L f(x)dx = 0$. That's why we can choose $\mathbf{v_N = 0}$.

The continuous spectral problem has eigenvalues

$$\lambda_k = \left(\frac{2\pi k}{L}\right)^2$$

and orthonormal eigenfunctions

$$w^k = \sqrt{\frac{1}{L}}\exp\left(\frac{2\pi ikx}{L}\right)$$

By transforming the f in the form

$$f(x) = \sum_{k=-\infty}^{+\infty} b_k w^k(x), b_k = (w_*^k, f),$$

we can express the exact solution in form of complex series

$$u(x) = \sum_{k=-\infty}^{+\infty} a_k w^k(x), a_k = \frac{b_k}{\lambda_k}$$

where $a_0 = b_0 = 0$.

When building DSES in matrix $A=WDW^*$ diagonal matrix D elements $d_k=\mu_k$ are replaced with eigenvalues of the continuous problem in such way

1) $d_k = \lambda_k$ when k = 1, 2, … , N/2,
2) $d_k = \lambda_{N-k}$ when k = N/2, …, N-1, $d_N = 0$.

Periodic function f (x) has the complex projection, which can be converted into real form

$$f(x) = \sum_{k=-\infty}^{\infty} a_k w^k(x) = \sum_{k=1}^{\infty} a_k w^k(x) + a_{-k} w^{-k}(x) + a_0 \frac{1}{\sqrt{L}} = \sum_{k=1}^{\infty} b_k \cos\frac{2\pi kx}{L} + c_k \sin\frac{2\pi kx}{L} + \frac{b_0}{2}$$

where $b_k = \frac{2}{N}\sum_{j=1}^{N} f_j \cos\frac{2\pi kj}{N}$, $c_k = \frac{2}{N}\sum_{j=1}^{N} f_j \sin\frac{2\pi kj}{N}$. For periodic vector $b_0 = 0$.

Thus, algebraic systems or DS-2 solution (vector) y may be made discrete Fourier series

$$y = \sum_{k=1}^{N/2} a^1{}_k \cos\frac{2\pi kj}{N} + a^2{}_k \sin\frac{2\pi kj}{N} + \frac{b_0}{2}$$, kur $a^1{}_k = b_k/\mu_k$, $a^2{}_k = c_k/\mu_k$.

Replacing the discrete eigenvalues μ_k including the N / 2 consecutive eigenvalues λ_k we get the Fourier series with N / 2 members (summands), which coincides with DSES algorithm.

These formulas can be easily used in analytical transformations, solving boundary problems with periodic boundary conditions. Mixed problems for partial differential equations, when using the method of straight lines, can be reduced to systems of ordinary differential equations, which can be solved analytically using the above-described spectral method and DSES.

Building higher-order DS-p with periodic conditions, we look at approximations derivative u'' (xj) using smooth grid xj = jh with p +1 point stencil (xj-p / 2, ..., xj-1, xj, xj +1 , ..., xj + p / 2). By using the method of undetermined coefficients with unknown coefficients Ck, Ep we look at approximation of order O (hp) in the form:

$$u''(x_j) = 1/h^2 \sum_{k=-p/2}^{p/2} C_k u(x_{j-k}) + E_p h^p u^{(p+2)}(\psi_j)/(p+2)!, \quad x_{j-p/2} < \psi_j < x_{j+p/2}.$$

We get such coefficients:

1) p=2: $C_1=1, C_0=-2, E_2=-2$,
2) p=4: $C_1=4/3, C_2=-1/12, C_0=-5/2, E_4=8$,
3) p=6: $C_1=3/2, C_2=-3/20, C_3=1/90, C_0=-49\backslash 18, E_6=-72$, etc.

Matrix A is in circulant form

$A=1/h^2\,[C_0, C_1, \ldots, C_{p/2}, 0, \ldots, 0, C_{p/2}, C_{p/2-1}, \ldots, C_2, C_1]$

with eigenvalues

1) p=2: $\mu_k=4/h^2\sin^2(\pi k/N)$,
2) p=4: $\mu_k=4/h^2(\sin^2(\pi k/N) + 1/3\sin^4(\pi k/N))$,
3) p=6: $\mu_k=4/h^2(\sin^2(\pi k/N) + 1/3\sin^4(\pi k/N) + 8/45\sin^6(\pi k/N))$, etc.

General formulas for DS-p approximation of order $O(h^p)$ are given in [1,13,33].

A matrix can be rewritten in the form A = WDW *, where eigenvectors remain (matrix W), but diagonal matrix contains the corresponding eigenvalues of the approximation sequence. This allows to get algorithms of a higher approximation DS-p, p = 2,4, ... which can be compared to DSES.

Similarly DS-2 and DSES algorithms are created for 2D Poisson, heat transfer, and wave-equation models, the result of using the method of lines, Cauchy problem of ordinary differential equations system. In the linear case analytical expressions may be obtained, but nonlinear problems can be easily solved numerically with MATLAB.

1.2. Problems with Dirichlet boundary conditions [15].

Many mathematical physics problems with Dirichlet boundary conditions must be solved, which can be easily transformed into a homogeneous problems. When building DSES will look at the previously used example: 1D Poisson equation with Dirichlet boundary conditions:

$$-u''(x) = f(x), \quad u(0) = u(L) = 0.$$

Solution for this problem

$$u(x) = \int_0^x (t-x) f(t)\,dt - \frac{x}{L}\int_0^L (t-L) f(t)\,dt$$

can be found using spectral method.

Continuous spectral problem has eigenvalues

$$\lambda_k = \left(\frac{\pi k}{L}\right)^2$$

and orthonormal eigenfunctions $w^k(x) = \sqrt{2/L}\sin\frac{\pi kx}{L}$.

By expressing the function f(x) in form $f(x) = \sum_{k=1}^{\infty} b_k w^k(x)$,

where $b_k = (f, w^k) = \int_0^L w^k(t) f(t)dt$.

We can express the exact solution in Fourier series:

$$u(x) = \sum_{k=1}^{\infty} a_k w^k(x), \quad a_k = \frac{b_k}{\lambda_k}.$$

Discrete problem DS-2 with N-1 algebraic equations can be rewritten in the form

$$Ay = f$$

where matrix A is standard 3-dioganal matrix with elements $1/h^2$ {-1, 2, -1}, y, f are column-vectors of order N-1. From the discrete spectral problem follows that eigenvalues are

$$\mu_k = \frac{4}{h^2}\sin^2\frac{k\pi}{2N},$$

and components of eigenvectors w^k can be expressed in form

$$w_j^k = \sqrt{2/N}\sin\frac{\pi kj}{N}.$$

Scalar product of orthonormal vectors w^k un w^m equals with Kronecker delta $\delta_{k,m}$. By placing these vectors in matrix W of order N-1 we gain WW = W^2 = E – identity matrix. After matrix property, matrix A can be expressed in the form $A = WDW^*$, where $W^* = W$.

By using this expression DS-2 or problem $Ay = f$ transform in the form $Dv = W^*f$, where $y = Wv$. For all $j = \overline{1, N-1}$ we can find coordinates of vector v:

$$v_j = \frac{1}{\mu_j}(W^*f)_j$$

By building DSES from DS-2 in matrix A=WDW diagonal matrix D elements $d_k = \mu_k$ are replaced with continuous problem's first N-1 eigenvalues λ_k.

$$f(x) = \sum_{k=1}^{\infty} c_k \sin\frac{\pi kx}{L}, \text{ tad } \quad c_k = \frac{2}{L}\int_0^l f(t)\sin\frac{\pi kt}{L}dt.$$

Thus by expressing u(x) in Fourier series we gain

$$u(x) = \sum_{k=1}^{\infty} a_k \sin\frac{\pi kx}{L}, \text{ where } a_k = c_k/\lambda_k.$$

This is actual also for discrete case because for vector of order N-1 with components f_j we have

$$f_j = \sum_{k=1}^{N-1} c_k \sin\frac{\pi kj}{N}, \text{ where } c_k = \frac{2}{N}\sum_{j=1}^{N} f_j \sin\frac{\pi kj}{N}.$$

That's why components of solution vector y (DS-2) can be expressed in discrete Fourier series

$$y_j = \sum_{k=1}^{N-1} a_k \sin \frac{\pi k j}{N},$$ where $a_k = c_k/\mu_k$.

Replacing the discrete eigenvalues μ_k including the N-1 continuous eigenvalues λ_k we get the Fourier series with N-1 members (summands), which matches with DSES algorithm.

DSES method is more stable compared to the DS-2, as the eigenvalues are larger $\lambda_k > \mu_k$.

These formulas can be easily used in analytical transformations, solving problems with Dirichlet boundary conditions. Mixed type problems of partial differential equations by using the method of lines can be reduced to systems of ordinary differential equations, which can be solved analytically using the above-described spectral method for creating DSES algorithms [15].

1.3. Problems with Robin boundary conditions [4,8,17].

DSES can be created for problems with Robin boundary conditions as well. Here are known difficulties when solving the spectral problems, because eigenvalues can be found only numerically from the corresponding transcendental equations. Eigenvalues and eigenvectors can be written analytical formulas depending on numerically obtained eigenvalues, but these are not matching in the points of homogenous grid. This might result in some inaccuracies when DSES are created and schemes might be not so efficient as in case with Dirichlet boundary conditions. Difficulties are raised in finding of the last discrete eigenvalues μ_N , μ_{N+1}, because the corresponding transcendental equations in the literature are not correct (see A. Samarsky „Difference Scheme Theory", 1977, in Russian). That's why these equations had to be recreated [17].

Similarly as in previous cases we look into Poisson's equation with Robin boundary conditions in the following form:

$- u''(x) = f(x)$, $u'(0) - \sigma 1\ u(0) = 0$, $u'(L) + \sigma 2\ u(L) = 0$, $\sigma 1 > 0$, $\sigma 2 > 0$.

The corresponding discrete algorithm (DS) is:

$-2(y_1 - y_0)/h^2 + 2\sigma 1\ y_0/h = f(0)$, $-(y_{j+1} - 2y_j + y_{j-1})/h^2 = f(x_j)$, $j=1,\dots, N-1$,

$-2(y_{N-1} - y_N)/h^2 + 2\sigma 2\ y_N/h = f(L)$,

where $h=L/N$, $x_j = j h$.

Further we get difference equations in form of algebraic system (DS-2)

$$Ay=f$$

with corresponding 3-diagonal matrix A of order M=N+1 with elements on the first row is $1/h^2\ \{2+2h\sigma 1, -2, 0\dots0\}$, the next one $1/h^2\ \{-1, 2, -1\}$ and the last – $1/h^2\ \{0,\dots, 0, -2, 2+2\sigma 2\}$, where y, f are column-vectors of order N+1 with elements y_j , $f(x_j)$, $j=0,\dots, N$.

By using scalar product of 2 vectoes y1 and y2

$[y1,y2]=h \sum_{j=1}^{N-1} y1_j\, y2_j + 0.5\, h\, (y1_0\ y2_0 + y1_N\ y2_N)$ we can show that $[Ay,y] > 0$.

For the corresponding discrete spectral problem $Aw^k = \mu_k w^k$, $k=1,\dots,N+1$

such soution exists (A. Samarsky, 1977)

$w^k = 1/C_k\ (1/\sqrt{2}\, w_0^k, w_1^k, \dots, w_{N-1}^k, 1/\sqrt{2}\, w_N^k)^T$,

$\mu_k = 4/h^2 \sin^2 (p_k h/2)$,

where $w_j^k = \sin(p_k h)/h \cos(p_k x_j) + \sigma 1 \sin(p_k x_j)$, $j=0,\dots,N$ is eigenvector w^k components, p_k is positive roots for these transcendental equation:

$\cot(p_k L)=(\sin^2 (p_k h)- h^2 \sigma 1 \sigma 2)/(h (\sigma 1 + \sigma 2) \sin(p_k h))$, $k=1,\ldots, N+1$.

Solution can be expressed in Fourier series form

$$u(x) = \sum_{k=1}^{\infty} a_k w^k(x), \quad f(x) = \sum_{k=1}^{\infty} b_k w^k(x),$$

where w^k (x) is orthonormal eigenfunctions, $a_k = b_k / d_k$, $b_k =(f, w^k)$, $d_k = \lambda_k{}^2$.
For discrete DS-2 solution

$$y = \sum_{k=1}^{N+1} a_k w^k, \; f = \sum_{k=1}^{N+1} b_k w^k, b_k = [f, w^k], \; a_k = b_k / d_k, d_k = \mu_k.$$

DSES and DS-2 matrix A is represented in form $A=WDW^T$, where D is diagonal matrix D with first N+1 eigenvalues of continuous problem $d_k = \lambda_k$, $k=1,\ldots,N+1$. If $d_k = \mu_k$ we get DS-2 with 3-diagonal matrix A.
If $\sigma 1= \sigma 2=\infty$ we gain problem with Dirichlet boundary conditions.
For mixed type problems and boundary value problems of partial differencial equations, by using the method of lines, can be reduced to ordinary differencial equation system which can be solved analytically using spectral method mentioned above, by creating DSES algorithms [17].

2. Hyperbolic heat conduction equation mathematical modeling of steel quenching process [5,8,17].

Steel quenching process is modelled actively in Latvia and the USA, which was proposed to be described as hyperbolic heat conduction equation by prof. Andris Buikis in 2005. The idea has become the core of international studies not only in Latvia but also with the participation of U.S., Ukrainian, Croatian and Greek scholars.

Steel hardening mathematical model – a mixed problem of hyperbolic heat conduction equation was solved numerically and analytically using the DS-2 and the DSES methods which helps to describe the intensive steel quenching in salty water, not oil, as it is done in classical hardening situation. H. Kalis together with A. Buikis by simulation of steel quenching process and by solving direct and inverse problems of hyperbolic heat conduction equation had led to results that triggered the interest of foreign physicists [5].
Mixed type problem for steel plate case is in the following form:

$$\tau \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}, \; x \in (0, L), \; t \in (0, t_f)$$

$$\frac{\partial T(0,t)}{\partial x} - \alpha_0 (T(0,t) - T_l) = 0, \frac{\partial T(L,t)}{\partial x} + \alpha_1 (T(L,t) - T_r) = 0,$$

$$T(x,0) = T_0(x), \; \frac{\partial T(x,0)}{\partial t} = V_0(x).$$

Here T is temperature, κ is

Here T is the temperature, κ is the thermal conductivity coefficient, α_0, α_1 heat transfer coefficients with the external environment with the temperatures T_l, T_r, τ is the relaxation factor, which turns parabolic heat conduction equation into the hyperbolic when it differs from a zero value. In practice, the heat value of the velocity at the beginning $V_0(x)$ is unknown, but the known are initial temperature $T_0(x)$ and the final temperature $T(x, t_f) = T_b(x)$. Therefore, to find the temperature distribution

T(x, t) and $V_0(x)$ need to solve inverse problem, which is ill posed, i.e. improper. Analytical solution for the problem in the form of final (discrete) Fourier series when method of lines, spectral projections DS-2 and DSES are used, allows to find the series' coefficients for function $V_0(x)$ and find the temperature distribution in the steel plate. Similarly the problem can be solved in the sphere and the sphere with a hole in the case of radial symmetry. In the 3D case (a piece of steel in the form of a polyhedron) it is possible to reduce problem to the problem of the plate with A. Buikis conservative averaging method. Pictures (1., 2.) show the maximum temperatures depending on the relaxation time at different rates $\tau = 0.1$, $\tau = 0.5$, where $t_f = 1$
$T_b(x) = T_l = T_r = 0$, $L = 1$, $T_0 = 600$, $\alpha_0 = 0$, $\alpha_1 = 0.071$. Accordingly we obtain
$V_0 = -6000.25$, $V_0 = -1387.8$.

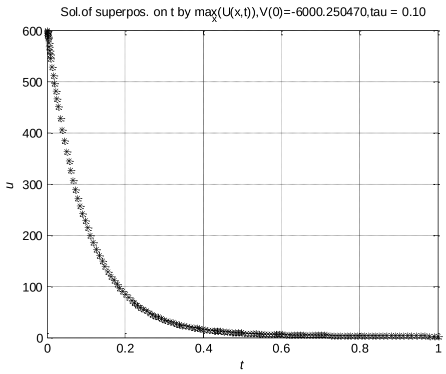


Figure 1(τ=0.1)

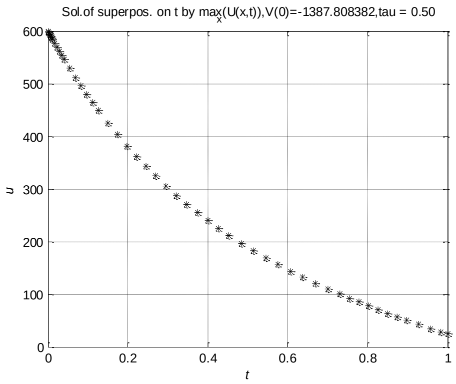


Figure 2 (τ=0.5)

To test the effectiveness of DSES two problems are solved – the wave equation (without an additive $\frac{\partial T}{\partial t}$, $\tau = 1$) and hyperbolic heat conduction equation ($\tau = 0.1$) with Dirichlet boundary conditions ($\alpha_0 = \alpha_1 = \infty$) $T_l = 1$, $T_r = 0$, $T_0(x) = V_0(x) = 0$. Figures 3, 4 show the jump of functions $T(x, t_f)$ at the time $t = 0.2$, solving the wave equation with DSES and DS-2, where $N = 30$ (see DS normal fluctuations in the field).

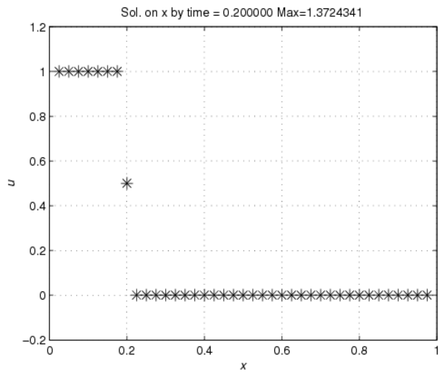


Figure 3 (τ=1, DSES)

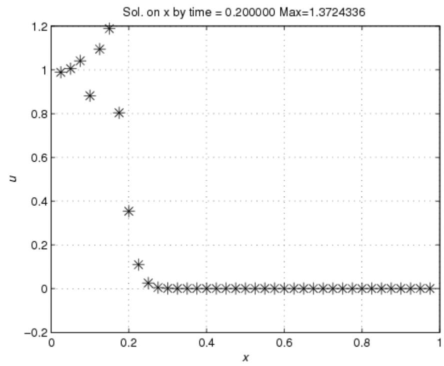
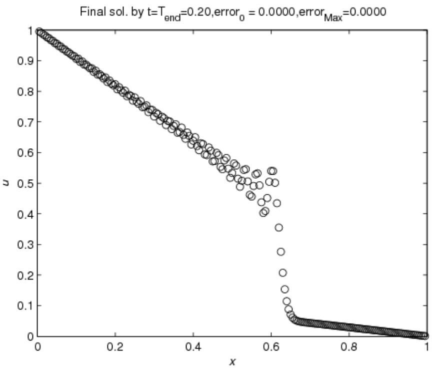


Figure 4 (τ=1, DS-2)

Similarly in figures 5 and 6 the results are shown for solutions of hyperbolic heat transfer equation with DS-2 and DSES when N=200 (can see the DS-2 oscillations in the jump point).



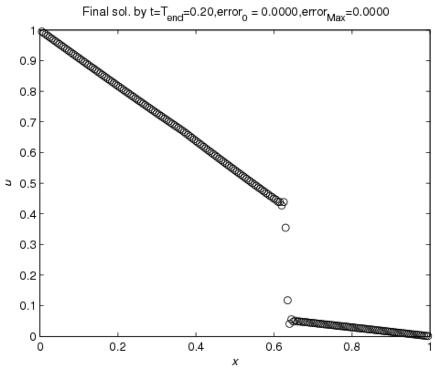


Figure 5 (τ=0.1, DS-2) **Figure 6 (τ=0.1, DSES)**

In [17] results are obtained for perforated ball tempering also. It shows DSES advantages over method with Fourier series (see for additional examples in [17]).

3. New type power plant mathematical modeling [2].

Transformers, motors and generators can be described by a common feature of the electron flow in electromagnetic systems. Analog connections are found in liquids, gases, whirlwinds in a storm and other streams. These vortices distributions can be used for a new type of power plant (J. Schatz 2003, I.Rechenberg 1988). In hydrodynamics (ideal fluid) velocity field dependence of the vortex field is characterized similarly as in electrodynamics – Biot-Savart law. In this approximation the different vortex (circular, spiral, etc.) affect to the speed of distribution inside the pipe, duct and cone was calculated (A.Buiķis, H.Kalis, J. Schatz 2005, 2006).

Actual and interesting problem is the heating of buildings with a clean, compact, effective devices. One of the earliest forms of modern applications is the transformation of AC electrical energy into heat energy. In [2] the authors study a single mathematical model of the device. It's a long cylinder filled with incompressible viscous fluid squeezing (the electrolyte), which is parallel to the cylinder axis placed metal rods (electrolyte), which is supplied AC with different phases (see the actual device with 6 electrodes in Figure 7). Mathematical model assumes that the cylinder of radius R is infinitely long, the electrodes are placed in the inner cylinder of radius r_0 inside and the 2D temperature distribution and fluid flow between two coaxial cylinders of cross-section is investigated (see device model in Figure 8). In the previous works of A. Buikis and H. Kalis (CMAM, 2002, 2(3), p. 247-251; MHD,2004,40(1), p.77-90; MMA, 2005, 10(1), p. 9-18) heat generators were modelled wit 6 and 9 circle-form electrodes which were wound around the cylinder surface. 3D flow in finite cylinder was modelled using software „Fluent" in work A.Buikis, L. Buligins, H.Kalis, MMA,2009,14(1), p.1-9.

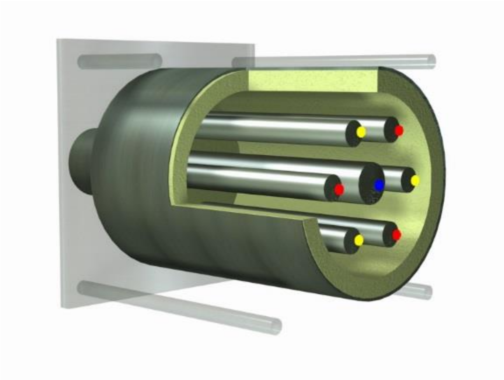


Figure 7 (real device)

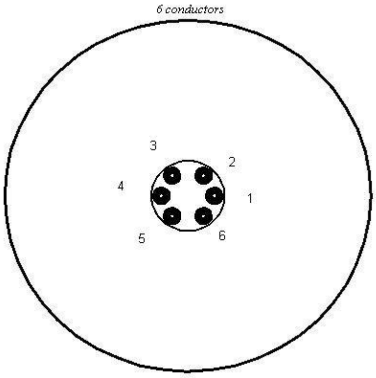


Figure 8 (model)

Stationary Navier–Stokes equation system in polar coordinates is used together with electromagnetic (Lorence) force which is caused by AC flow through the electrodes. By calculating the source members in hydrodynamics and heat conduction equations, excluding pressure and by introducing dimensionless temperature T(r, φ) and the hydrodynamic power ψ (r, φ) function problem is reduced to the problem of the border values of two partial differential equations with periodic conditions for the angle φ:

$\Delta^2 \psi = 1/r\, J(\Delta\psi, \psi) + K_H \langle f \rangle$,

$\psi(r,0) = \psi(r, 2\pi)$, $\dfrac{\partial \psi(r,0)}{\partial \varphi} = \dfrac{\partial \psi(r,2\pi)}{\partial \varphi}$,

$\psi(\eta,\varphi) = \psi(1, \varphi) = 0$, $\dfrac{\partial \psi(\eta,\varphi)}{\partial r} = \dfrac{\partial \psi(1,\varphi)}{\partial r} = 0$,

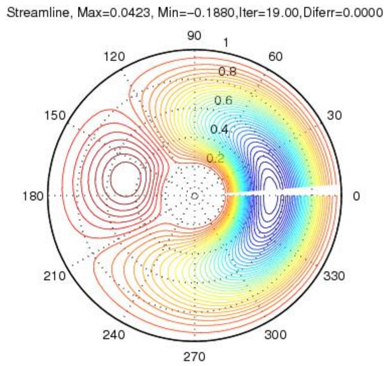
****

$\Delta T = Pr/r\, J(T,\psi) - K_T \langle j_z^{\,2} \rangle$,

$T(r,0) = T(r, 2\pi)$, $\dfrac{\partial T(r,0)}{\partial \varphi} = \dfrac{\partial T(r,2\pi)}{\partial \varphi}$,

$\dfrac{\partial T(\eta,\varphi)}{\partial r} = 1$, $\dfrac{\partial T(1,\varphi)}{\partial r} = 0$,

where K_H , K_T are source member parameters, <f> – averaged value of electrodynamics forces in time, $\langle j_z^{\,2} \rangle$ – averaged value of current density, Pr – Prandtl number, $\eta = r_0/R$ – proportion of cylinder radiuses, J – Jakobian, Δ – Laplace operator. An original methodology for calculating the source function was created. By using the lower relaxation method and operations of the circular matrices the algorithm is created. With its help you can determine the maximum cylinder temperature and vortex breakdown dependency on the number of electrodes (3, 6 or 9), position and shift of the AC phases. For example in case with 6 electrodes results are shown in Figure 9 (vortex breakdown) and 10 (temperature distribution with the maximum temperature of 10.56). This temperature was gained with phase shift 60° and electrode positioning [1,2,3,4,5,6] what is shown in Figure 8. If phase shift is 120°, maximal temperature is 8.22 units for electrode positioning [2,4,6,1,3,5].



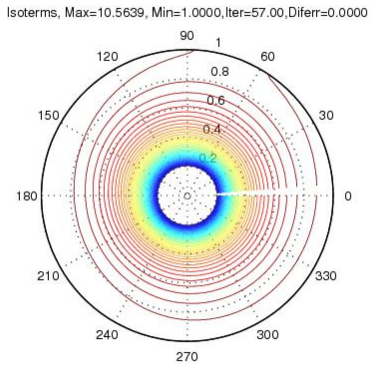


Figure 9 (vortex breakdown) **Figure 10 (temperatures)**

4. Mathematical modelling of metal particles in peat layers [14,16,24].

By simulating three peat layers metal Ca and Fe concentrations in peat layers were studied, solving 3D boundary value diffusion equation with piecewise constant coefficients in layers. In the difference scheme DS-2 created with the periodic conditions in the given direction, circular matrix algorithms are used, which simplifies the calculations. Numerical results are compared to experiments. Before this 2 layers of peat were modelled by H.Kalis, E.Teirumnieks, E. Teirumnieka, I.Kangro in work „Proc. of the 7-th int. scientific practical conference"Environment. Technology. Resources.",Rezekne higher education institution, June, 2009, p. 249-253.

Metal diffusion process in peat has been modelled in 3D parallelepiped

$\Omega = \{ (x,y,z) : 0 < x < l, 0 < y < L, 0 < z < Z \}$.

Region Ω consists of N layers. Stationary 3D diffusion problem is modelled with piecewise constant coefficients in the layers D_x, D_y, D_z.

$\Omega_i = \{(x,y,z): x \in (0,l), y \in (0,L), z \in (z_{i-1}, z_i), i=1,\dots,N\}$,

where $H_i = z_i - z_{i-1}$ is height of the layer, $z_0 = 0$, $z_N = Z$.

Must find the concentration $c_i = c_i(x, y, z)$ distribution in each layer Ω_i, by addressing the border value problem for partial differential equations (PDE) - diffusion equation:

$$D_{ix} \frac{\partial^2 c_i}{\partial x^2} + D_{iy} \frac{\partial^2 c_i}{\partial y^2} + D_{iz} \frac{\partial^2 c_i}{\partial z^2} = 0,$$

$c_i = c_{i+1}$, $D_{iz} \frac{\partial c_i}{\partial z} = D_{(i+1)z} \frac{\partial c_{i+1}}{\partial z}$, when $z = z_i$, $i=1,\dots,N-1$;

$c_i(0,y,z) = c_i(l,y,z)$, $\frac{\partial c_i(0,y,z)}{\partial x} = \frac{\partial c_i(l,y,z)}{\partial x}$, periodic conditions in x direction,

$\frac{\partial c_i(x,0,z)}{\partial y} = \frac{\partial c_i(x,L,z)}{\partial y} = 0$, symmetry conditions in y direction,

$\frac{\partial c_1(x,y,0)}{\partial z} = 0, c_N(x,y,Z) = C_a(x,y)$,

where $C_a(x, y)$ is given concentration distribution.

Problem with 3 layers (N=3, see figure 11) is solved using A. Buikis implemented constervative averaging method and DS-2 methods, by reducing the 3D problem to 2D PDE system with circular matrices in x direction. The ratio of diffusion coefficients is determined from the 1D diffusion equation z-direction analytic solution on the basis of experimental measurements of peat interlayers. In figure 12 concentration breakdown can be seen in z direction, other results are in [16].

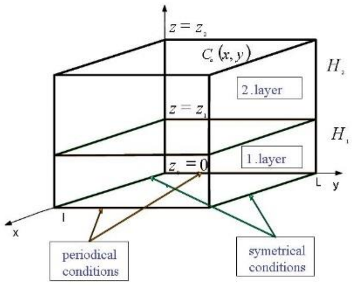


Figure 11 (3 peat layer scheme)

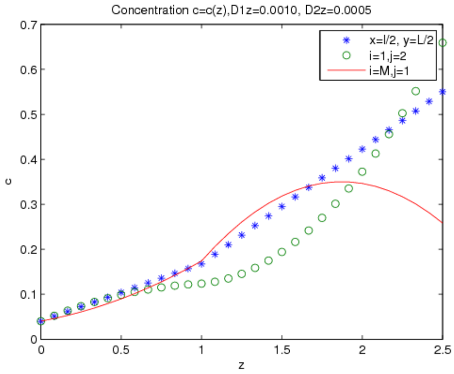


Figure 12 (concentration in z direction for 3 fixed points)

5. Magnetized elongated droplet and thread motion simulation in rotating magnetic field [9,10,11,18,19,26,29]

In the result of fruitful cooperation with the physics professor. A.Cebers 3 articles [9,18,19] has been published international journals and article [29] was submitted for publication on the ferromagnetic particles – elongated droplet motion in external rotating magnetic field, using the droplet dynamics and curvature equations. Process is described in 2D nonlinear parabolic-type partial differential equations, which are handled by the numerical method of lines, replacing the second or the fourth order derivatives of the spatial variable with the 2nd order approximation of the finite differences. Equations of the second order derivatives are formed using DS-2, DSES, and ordinary differential equation systems at time t with equation count of 100 - 200, carried out with the software MATLAB.

5.1. Modeling of elongated ferromagnetic droplet motion in external rotating magnetic field [10,19,29]

Stretched droplet motion in external rotating magnetic field is described by non-linear parabolic equation with a variable diffusion coefficient signs (see Figure 13 where diffusion coefficient F 'is the derivative of the function F). Since the values of this coefficient equals with zero in some places (or even negative), the received solution is with jumps. The solution can be obtained by modifying the equation by introducing additional summand of regularization, which contains higher-order partial derivatives. Regularization was got by replacement of non-monotonic sign changing diffusion coefficient F 'with a modified positive function (see Figure 14). Using DSES and MATLAB solver "ode15s" we managed to calculate this problem directly without regularization. From the results different magnetic droplet configurations and shapes are drawn (S-shaped, 8-shaped and spiral).

Stretched droplet motion is described with the following dimensionless differential equation

$$\frac{\partial \beta}{\partial t} = \frac{\partial^2}{\partial l^2} F(\beta) - \varepsilon \frac{\partial^5 \beta}{\partial l^4 \partial t} + \omega\tau \ ,$$

where F(β) = 1/Bm β + sin(2 β) is nonlinear function (Figure 13), β(l, t) is the shift of phase, what is formed by the local tangent of center line of the droplet and abscissa axis, ω is the angular frequency, τ is a time scale factor, ε is a small factor (circa 10^{-4}), Bm is the magnetic Bond number, l is the arc length, 0 < l < L, t – time, 0 < t < t_f. Regularization member with ε is added from physics consideration. Function F(β) is not monotonic when Bm > 0.5. Equation comes together with boundary conditions

$$\beta(0,t) = \beta(L,t) = \frac{\partial^2 \beta(0,t)}{\partial l^2} = \frac{\partial^2 \beta(L,t)}{\partial l^2} = 0$$

and initial condition $\beta(x,0) = \beta_0(x)$ where $\beta_0(x)$ is a given function (usually $\beta_0(x) = 0$ – initial form of droplet is straight).
When ε = 0 we get ill posed problem. Further we look into droplet with length L = 2.

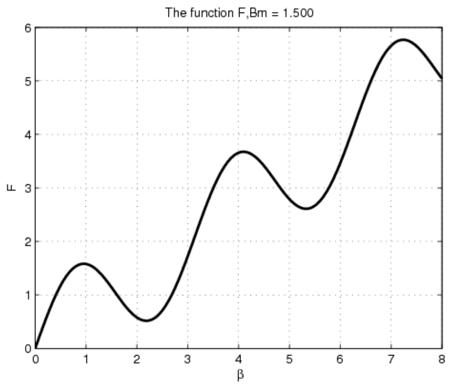


Figure 13 (function F)

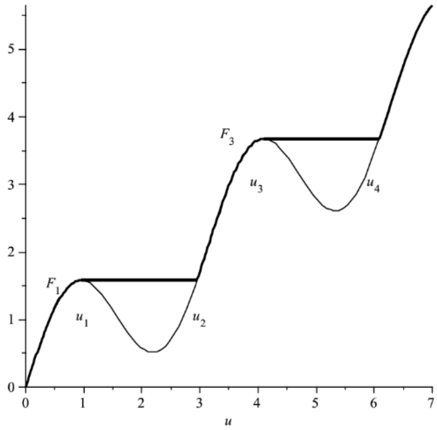


Figure 14 (modified function F)

The form of droplet we get by integrating these equations

$$\frac{dx}{dl} = \cos(\beta), \frac{dy}{dl} = -\sin(\beta).$$

The integration constants are determined from the condition that the droplet center of mass remains stationary.

Stationary solution $\beta_s(l)$ is found using boundary conditions $\beta_s(0) = \beta_s(2) = 0$ from the transcendental equation in form $F(\beta_s(l)) = 0.5 l(2-l)\omega\tau$. Maximal value of β_m we get from the equation $F(\beta_m) = 0.5\omega\tau$. Solution $\beta(l,t) > 0$ is symmetrical according to l=1. Angle β as function dependent on arch length l is not continuous when ωτ > 2 F(β_0) when β_0 is root of equation F'(β)=0 (maximal value of function F). Value $w_c = (\omega\tau)_0 = 2$ F(β_0) defines the critical frequency. Figures 15 and 16 show stationary solution with 3 jumps (ωτ=15) and one jump (ωτ=5).

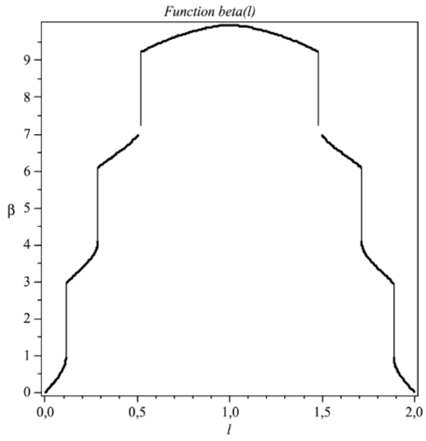


Figure 15 (stationary solution ωτ=15)

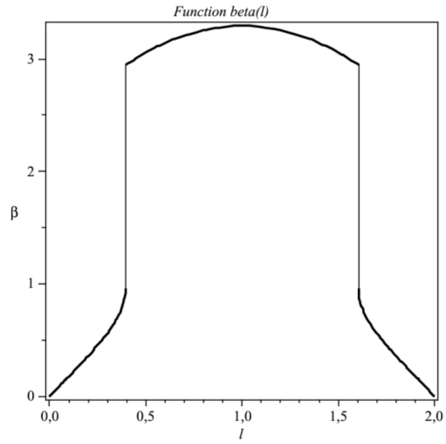


Figure 16 (stationary solution ωτ=5)

Modified function F (Figure 14) is monotone with derivatives with limited values $0 < F'(\beta) < 2 + 1 / Bm$ and we can use approximate assessments for solution of the equation with $\varepsilon = 0$ and it belongs to Sobolev space for each fixed time unit t.
The given problem is solved with MATLAB software, by using the method of lines and finite difference methods in space and approximating spatial derivatives with central differences with approximation of the second order (DS-2).

We inspect homogenous grid in space $l_j = jh,\ j = 0,\ldots, N,\ Nh = 2$.
We obtain the initial values of the discrete problem in the form of system of N-1 ordinary differential equations

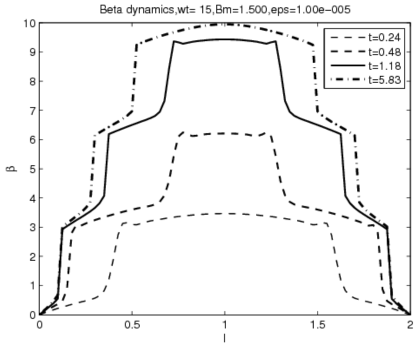
$$(E+\varepsilon B) \frac{dU(t)}{dt} + A\, F(U(t)) = G, \quad U(0)= U_0 ,$$

where E is identity matrix, A is standard 3 diagonal matrix of order N-1 with elements $1/h^2\{-1;2;-1\}$, which approximate the second order derivative $-\frac{\partial^2}{\partial l^2}$,
$B=A*A$ is 5-diagonal matrix of order N-1 with elements $1/h^4\ \{1;-4;6;-4;1\}$ which approximates the derivative of the 4th order $\frac{\partial^4}{\partial l^4}$,
U(t), U_0, F(U), G are column-vectors of the order N-1 with elements
$u_j(t) \approx \beta(l_j, t)$, $f_j(u) \approx F(u_j(t))$, $g_j = \omega\tau$, $j = 1,\ldots, N-1$.
Differential equation can be expressed in the form

$$\frac{dU(t)}{dt} = (E+\varepsilon B)^{-1} (G - A\, F(U(t))).$$

By building difference scheme with exact spectrum (DSES) matrices A and A^2 are replaced by their spectral representations WDW and WD^2W, where $W = W^{-1}$ is a symmetric orthogonal matrix with elements $w_{j,k} = \sqrt{2/N} \sin(\pi jk / N)$, $j, k = 1,\ldots,N-1$, and the diagonal matrix D contains N-1 eigenvalues of the differential operator $-\frac{\partial^2}{\partial l^2}$ which are in form $d_k = (\pi k / 2)^2$ (ordinary difference scheme has $d_k = 4/h^2 \sin^2(0.5\, \pi k / N)$).

Numerical results were obtained using Matlab, when Bm = 1.5, $\omega\tau$ = 5, 8, 15, t_f = 3, 5, 6, ε = 0, 10-4. Moment of time t_f is the time when the non-stationary solution has converged to stationary solution. With the increase of frequency droplet gets more spiral shape. Figures 17 and 18 show solution and the droplet shape dynamics, when $\varepsilon = 10^{-4}$, $\omega\ \tau = 15$.



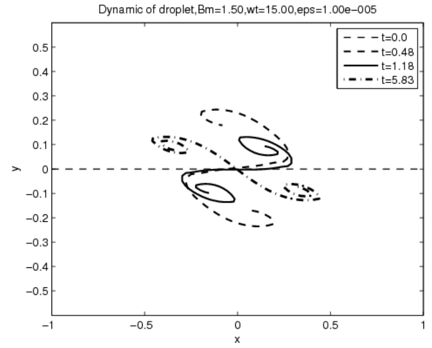


Figure 17 (β dynamics) **Figure 18 (dynamics of form)**

In [29 hysteresis phenomenon] is modeled, which occurs by changing the frequency of the field.

5.2. Motion modeling of ferromagnetic elastic thread (bar) in external rotating magnetic field, the dynamics and curvature equations [9,11,18]

Let's denote partial derivative $\frac{\partial q(l,t)}{\partial l}$ of function q(l, t) with q'(l, t). By using expressions of force and momentum, flexible bar velocity equation $\mathbf{v}(l,t) = \frac{\partial r(l,t)}{\partial t}$ in 2D thread section can be written in form

$\zeta \mathbf{v} = -A_0\, \mathbf{r}^{(4)} - M\, \mathbf{n}' + (\tilde{\Lambda}\ \mathbf{t})'$,

where r = (x, y, 0) are 2D vector coordinates, **t, n** are the vectors of center line tangent and the normal of the elastic thread,

t=(x', y',0), **n**=(-y', x',0),

$\mathbf{r}^{(4)}$ is partial forth order derivative of variable l,

$\tilde{\Lambda} = \Lambda - 1.5\, A_0\ K^2$, K=K(l,t) is center line curvature,

Λ = Λ (l,t) are Lagrange multipliers,

A_0 is curvature modulus,

ζ is the hydrodynamic drag coefficient,

M is a dissipative torque for 1 length unit of thread.

To solve the equations of 3D thread dynamics, we introduce homogenous grid with N +1 points l_k and step h and arc variable l. From assumption that thread cannot be stretched, constraint in the form of equation $g_k = (r_{k+1} - r_k)^2 = h^2$, k = 1, ..., N follows.

By introducing Jakobian matrix **J** of order N * 2 (N+1) with elements $\frac{\partial g_k}{\partial r_j}$ follows that **J**.**v**=0.

By using projection operation $P=E-J^T\ (J\ J^{\ T})^{-1}J$, (E is identity matrix) we discretize dynamics equations with central differences without summand $(\tilde{\Lambda}\ t)'$ in the form of ordinary differential equation system with N + 1 equations. It can be solved numerically with Matlab for various magneto-elastic number $Cm=M\ L^2\ /\ A_0$ [9]. Figure 19 shows dynamics of the forms when Cm = 300.

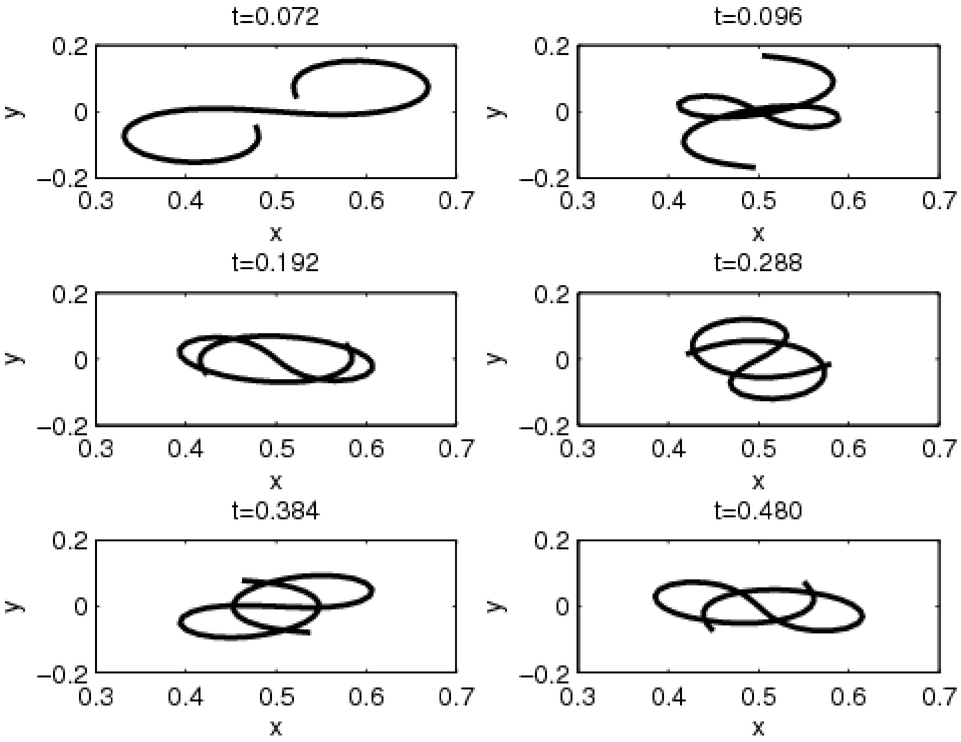


Figure 19 (dynamics of form, Cm = 300, t < 0.49)

For small values of Cm thread forms S-type form, which rotates around the center of mass in the direction of the applied torque. For large values of Cm loops are formed at the free ends of the thread (Figure 19), which rotates in direction of the applied torque in the stationary case.

Let us review the internal dynamics of the magnetic thread curves, using the curvature equations. Force vector **F** is in the following expression: $F = - A_0 r''' - M n + \tilde{\Lambda} t$.

From the Frenet formulas $t' = - K n$, $n' = K t$ and expressions

$\mathbf{t}=\mathbf{r}'$, $\mathbf{n}\ \mathbf{t}=0$, $\mathbf{nn}=1$, $\mathbf{v}= \mathbf{v}_n\ \mathbf{n} + \mathbf{v}_t\ \mathbf{t}$, ($v_n$, v_t are the normal and tangential velocity components), we get curvature equations in the following form

$$\frac{\partial K(l,t)}{\partial t} = -(\mathbf{v}_n'' + K^2\, \mathbf{v}_n) + \mathbf{v}_t\, K'.$$

By transforming it we get

$$\zeta\ \frac{\partial K(l,t)}{\partial t} = -\left(\frac{\partial^2 K(l,t)}{\partial l^2} + K^2\right)(A_0\, (K''+0.5\, K^3) - \Lambda\, K) + K'(\Lambda' - MK)$$

From the condition that the thread cannot be stretched

$\mathbf{v}_t' = - A_0\, K\, K'' - 0.5\, A_0\, K^4 + \Lambda\ K^2$, equation follows

$\Lambda'' - M\, K' + A_0\, (K\, K'' + 0.5\, K^4) - \Lambda\ K^2 = 0$

and thread curvature equation takes the form

$$\zeta\ \frac{\partial K(l,t)}{\partial t} = -A_0\, (K^{(4)} + 3.5\, K^2\, K'' + 3\, K\, (K')^2 + K^5) + 2\, \Lambda\ K^3 + 3\, \Lambda' K' + \Lambda\ K''\ .$$

Transforming the equations in dimensionless form we obtain two partial equations:

$$\frac{\partial K}{\partial t} = -K^4 - 3.5\, K^2 K'' - 3\, K\, (K')^2 - K^5 + 2\, \Lambda\, K^3 + 3\, \Lambda' K' + \Lambda\, K'',$$

$$\Lambda'' - Cm\; K' + K(K'' + 0.5\, K^3) - \Lambda\; K^2 = 0.$$

If both ends of the thread l = 0 and l = 1 are free, we get boundary conditions:
$K(0,t) = \Lambda(0,t) = K(1,t) = \Lambda(1,t) = 0$, $K'(0,t) = K'(1,t) = Cm$.
If only one thread end l = 0 is free, then $K(0,t) = \Lambda(0,t) = 0$, $K'(0,t) = Cm$,
$K'(1,t) = \Lambda'(1,t) = K''(1,t) = 0$.
The form of the thread is found from Frenet formulas
$x'' = K\, y'$, $y'' = -K\, x'$.

Stationary solution x = x (l), y = y (l) the initial condition at l = 0 (fixed end) is x(0) = y(0) = 0, y'(0) = 0, x'(0) = 1. In order to compare the dynamics of the thread, these conditions are chosen from the results in the fixed time, obtained by integrating directly the equations of motion. Good matching results are received (see Figure 20 when Cm = 50; results obtained integrating the equations of motion directly are drawn with solid lines, but dashed shapes shows forms derived from curvature equations). Figure 21 shows the stationary solution, which is obtained using values Cm = 9, 19, 29, 39, 49, 59. The calculations in Matlab space discretization in homogenous grid is used with $l_j = (j-1)\, h$, $h = 1 / N$, $j = 1, ..., N+1$, and a special finite difference scheme for approximation of Lagrange multipliers [9].

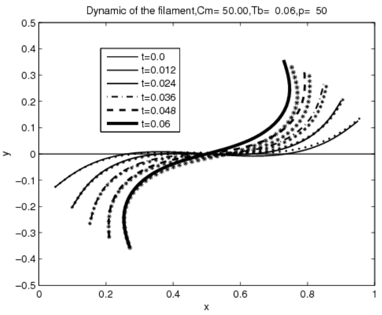


Figure 20

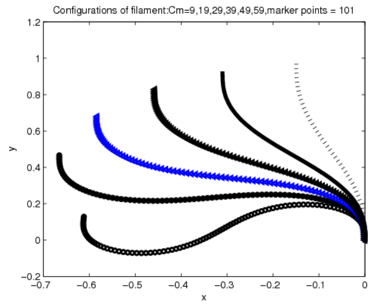


Figure 21

3. Modeling of nonlinear heat transfer [7]

In [7] we dealt with a non-linear heat transfer equation which arises in modeling of the "blow-up" dissipative 1D and 2D heat flows between two coaxial cylinders. These flows have been studied extensively in 1970–1990 by academician A.Samarsky and works of his scholars, for example in the book "Blow-up in Quasilinear Parabolic Equations" 1987, in Russian. We consider the problem of 2D two-layer medium between the two coaxial cylinders. Using the method of lines, the DS and the DSES with circulant matrices, the mixed type problem is reduced to the initial value problem of non-linear ordinary differential equations system, which is solved numerically by iterations to obtain the stationary solutions. 1D mixed problem in polar coordinates is in the following form:

$$\frac{\partial u(r,t)}{\partial t} = \frac{\partial}{r\partial r}\left(\lambda r \frac{\partial u^{\sigma+1}(r,t)}{\partial r}\right) + a u^{\beta}(r,t),\ r \in [r_0, R],\ t > 0,$$

$u(r_0, t) = U(R,t) = 0$, $u(r,0) = u_0(r)$,

where $\sigma \geq 0$, $\beta > 0$, $\lambda > 0$, $a \geq 0$ are constant parameters, $u_0(r) \geq 0$ is a given function. If environment is not homogeneous, but is composed of several layers of radius r direction, the coefficients λ, a can be different. In work of H.Kalis, I.Kangro, A.Gedroics, IJPAM, vol. 71, no. 1, 2009, 575-592, this problem is solved for a single layer in Cartesian coordinates. In [7] polar coordinates with radial symmetry are used, theoretical estimates are derived for time moment when the solution becomes stationary, blows up (in finite time interval the solution tends to infinity) for homogeneous and 2-layer mediums.

For 2D area $\{(r, \varphi, z): r_0 < r < R, 0 \le \varphi \le 2\pi\}$ with heat conductive material at a constant temperature u = 0 on the cylinder surfaces in layered medium, we get mixed type problem for temperature function $u = u(r, \varphi, t) \ge 0$, which satisfies the nonlinear heat transfer equation:

$$\frac{\partial u}{\partial t} = \lambda \Delta g(u) + af(u),\ r \in [r_0, R],\ t > 0, \varphi \in [0, 2\pi],$$

In each layer parameters λ, a are piecewise constant and conditions of continuously on the layer boundaries, Δ is the Laplace operator in polar coordinates, g(u) = u σ +1, f(u) = uβ.

In the discrete case for two layers ($\lambda_1 = 100$, $\lambda_2 = 1$, $r_0 = 0.2$, $R = 1$, $r_1 = 0.6$ – boundary position of layers), using Gaussian eliminating (factorization) method with circulant matrices, we obtain the stationary solution (Figure 22), for σ = 3, β = 4, a = 59.2001 = μ_1 (μ_1 is the first eigenvalue of the Laplace operator for two layers with radial symmetry) and "blow-up" solution (Figure 23), for σ = 3, β = 5 a = 500, which locally tends to infinity in a fixed r = 0.75 at time t = 32.44097 (figure shows two time moments T1 = 32.44095 and T2 = 32.44096 with doesn't differ much, but the temperature is changing very fast).

Theoretically and numerically is proved in the work that:

1) when $\beta < \sigma + 1$, the solution tends to zero for all $a > 0$,
2) when $\beta = \sigma + 1$, a < μ1, the solution tends to zero, $a = \mu_1$, then we obtain the stationary solution different from zero, $a > \mu_1$, then the solution tends to infinity globally,
3) when $\beta > \sigma + 1$, for sufficiently large values of *a* solution in some fixed point in time tend to infinity locally, i.e. forms blow-up solution.

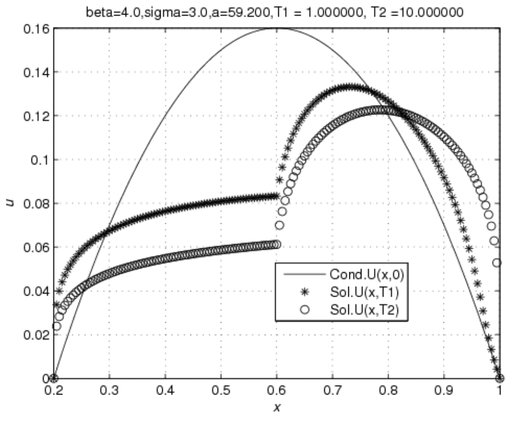


Figure 22 (σ = 3, β = 4)

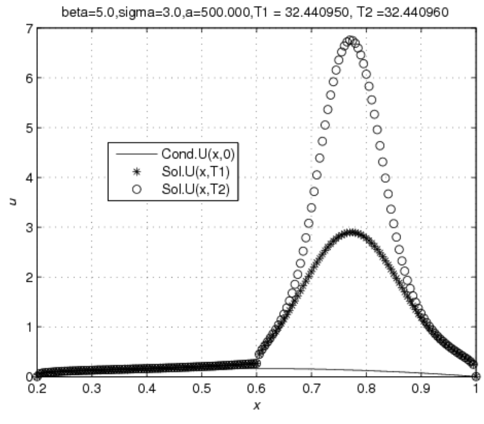


Figure 23 (σ = 3, β = 5)

7. MHD flow modeling [3,22,27,30,32]

When stirring an electricity conducting incompressible liquid in magnetic field, such as molten liquid metal, in technological applications it is important to know it's structure, vortex generation or loss depending on the parameters in advance.

In [3], the fluid is located between two infinite coaxial cylinders, the surfaces can be rotated. The unit is supplied with different type (homogeneous, radial, axial, dipolar) external magnetic field. Process can be described in the so-called non-inductive approximation when the induced magnetic field is ignored. Magnetohydrodynamic (MHD) flow which is caused by an external magnetic field in a cylindrical cross-section area is induced and it is calculated with the finite difference method using calculation algorithms for circulant matrices.

The external magnetic field builds radial $F_r(r, \varphi)$ and azimuthal $F_\varphi(r, \varphi)$ Lorentz force vector **F** components. Force **F** rotor axial vector component causes fluid motion with velocity vector **V** with radial $V_r(r, \varphi)$ and axial $V_\varphi(r, \varphi)$ components (Figure 24 with homogenous magnetic field), which is described by the stationary Navier-Stokes equations ring (r, φ): $r_0 < r < R$.

From 2D vector Lorentz force **F = J x B** in magnetic field with induction vector components $B_r(r, \varphi)$, $B_\varphi(r, \varphi)$ it can be obtained that $F_r = - \sigma B_\varphi(V_r B_\varphi - V_\varphi B_r + E_z)$, $F_\varphi = \sigma B_r(V_r B_\varphi - V_\varphi B_r + E_z)$ where E_z = const is the azimuthal component of electric field E, **J = σ (E + V xB)** current density vector, which is determined from Ohm's law, σ is the electrical conductivity. Cylinder surfaces r = R and $r = r_0$ rotates at a speed $V_\varphi = r_0 \Omega_0$ and $V_\varphi = R \Omega_1$, where Ω0, Ω1 are the corresponding angular velocities.

By excluding the the pressure from the Navier-Stokes equations and introducing the function of the current ψ in the expressions

$V_r = r^{-1} \frac{\partial \psi}{\partial \varphi}$, $V_\varphi = - \frac{\partial \psi}{\partial r}$ we get dimensionless partial differential of 4th order

$r^{-1} J(\Delta\psi, \psi) = Re^{-1} \Delta^2 \psi - S f$,

where f is force F rotor axial component,

J is the Jacobian of two functions, Δ – Laplace operator in polar coordinates,

$Re = U_0 R / \nu$ - Reynolds number, $S = \sigma B_0^2 R / (\rho U_0)$ – Stewart number (U_0, B_0 are velocity and magnetic induction characterizing properties, ν – kinematic viscosity, ρ – density of the fluid). By using finite difference and lower relaxation techniques, by calculating biharmonic operator with factorization (Gaussian elimination) method, numerical results are obtained depending on the magnetic field type and characteristics described by parameters Re and S. In figure 25 vortices can be seen among the cylinders in a homogenous magnetic field, where $r_0 = 0.2$, R = 1, Re = 100, S = 10, $\Omega_0 = 5$, $\Omega_1 = -1$ and the inner cylinder is paramagnetic with infinitely large relative permeability.

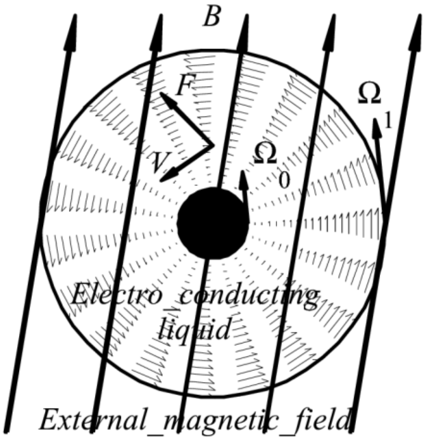


Figure 24 (scheme of device)

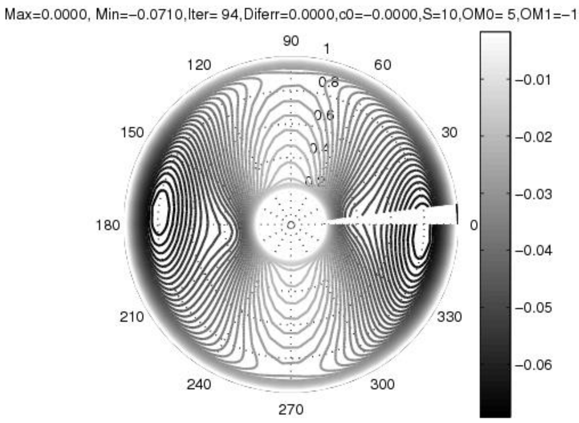
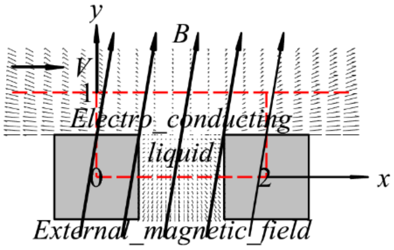


Figure 25 (cylinders rotate contrary)

In [30] an infinite number of quadratic cylinders positioned periodically are modelled which are flown around in 2D with a viscous incompressible electrically conductive liquid in homogenous magnetic field. We investigate two different cylinder positions: in parallel lines and 2 different shifted parallel rows. By following conditions of periodicity and symmetry, 1 section of the 2D region containing 2 cylinders are chosen for the calculation (Figure 26 and 27).



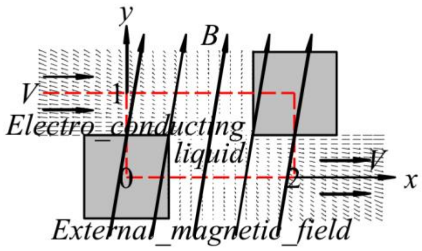


Figure 26 (on parallel line) **Figure 27 (on 2 parallel lines)**

2D dimensionless external magnetic field contains 2 induction vector components: $B_x = \cos(\alpha)$, $B_y = \sin(\alpha)$, where α is the angle between the Ox-axis and the induction vector. Various types of magnetic field are tested: parallel to the Ox-axis ($\alpha = 0$), transverse ($\alpha = \pi / 2$) and slope ($\alpha = \pi / 4$ and $\alpha = 3\pi / 4$). Magnetic field builds Lorentz force **F** two components

$F_x = - \sigma B_x (V_x B_y - V_y B_x + E_z)$, $F_y = \sigma B_x (V_x B_y - V_y B_x + E_z)$

where E_z = const is electrical field E axial component, V_x, V_y are the velocity vector **V** components. Vector rot **F** z-component causes fluid motion, which is described with Navier-Stokes equations in Cartesian coordinates.

When importing power function ψ and illuminating pressure we get system of two equations

$$0= J(\psi,\zeta) +Re^{-1} \Delta \zeta + S f, \zeta = - \Delta \psi,$$

where f is a z-component of vector rot F, J – Jacobian, ζ – function of the vortices.

Using the finite difference and lower relaxation techniques, results of calculations and several figures are given in the work [30] depending on Reynolds Re, Stewart S numbers and homogeneous field direction.

8. Electron flow modeling in gyrotron [12,20,21].

In works submitted by several authors are studied and numerically modelled complex nonlinear Schrödinger-type partial differential equations system that describes one or more electron RF field amplitude fluctuation modes f(x, t) in gyrotron and transversal orbital momentum p(x, t) depending on time t and x from segment [0, L]. Work [20] is designed as a review article of several years of research in Euratom and the recent developments in this direction.

We analyze two versions of the gyrotron resonators: new (after 2008) and the old one. High-frequency RF field amplitude f(x, t) of the resonator and the electron transversal orbital momentum p(x, t, θ) with parameter θ ($0 < \theta \le 2\pi$) can be described by a complex system of differential equations (the new version):

$$\frac{\partial p}{\partial x} + i\,(\Delta + |p|^2 - 1 - gb)\,p = if(t,x),$$

$$\frac{\partial^2 f}{\partial x^2} - i(1+\delta)\frac{\partial f}{\partial t} + (1+0.5(\delta + gc))gd\, f = (1+\delta)(1+gc)^2\;)\, I <p>,$$

where i is the imaginary unit, $0 \le x \le L$, $t \ge 0$ is the axial and time coordinates, L – interactive field length, Δ, δ, θ – real constants, I – current, gb(x), gc(x) , gd(x) – given real field variable x values, <p> averaged (integral) at θ variable p value. The system is accompanied by the initial conditions $p(t,0,\theta)=\exp(i\theta)$,$f(0,x)=f_0(x)$,

and boundary conditions $f(0,t)=0$, $\frac{\partial f(L,t)}{\partial x} = -i\gamma\; f(t,L)$,

where $f_0(x)$ is a given complex function, γ is positive parameter.
For the old version of gyrotron gb=bc=gd=0.

It turned out that the approximation in space with central differences (DS-2) and the implicit difference scheme is unstable for new version of gyrotron, because the solution by decreasing the time step has became oscillating in time and space as well (O. Dumbraja numerical experiments 2010). To find the solution's correspondence to physics, the method of lines was used, reducing the partial differential equations in time to system of ordinary differential equations and solving them with Matlab solvers, where the time step is selected automatically for the given precision. Numerical calculations showed that the stationary solution f fluctuations in the space really exists (Figure 28), but not in the time (see the other results of the work [21]).

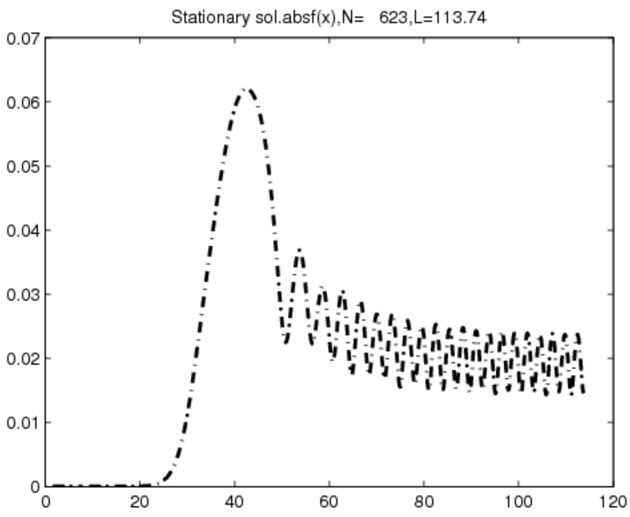


Figure 28 (stationary solution f)

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