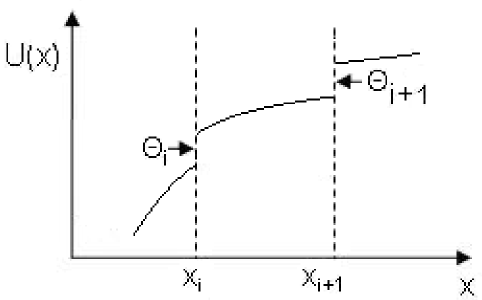
UNCONVENTIONAL USAGE OF CLASSIC METHODS OF MATHEMATICAL PHYSICS IN MATHEMATICAL MODELING

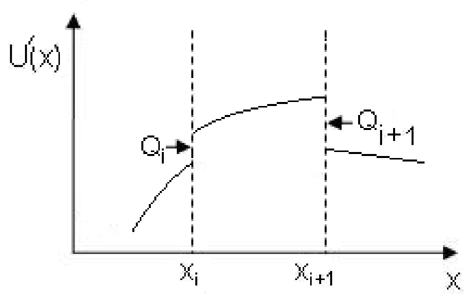
Dr. Andris Buiķis, Dr. Margarita Buiķe, Dr. Raimonds Viļums, MSc. Tabita Bobinska, MSc. Sanda Blomkalna, MSc. Marija Lencmane

1. General Information

About 30 years ago A. Buikis defined a new class of splines: the integral splines. Integral parabolic splines (IPS) firstly were defined in papers of A. Buikis [4] - [7]. IPS and accordingly integral rational spline was defined for continuous, pieces smooth class of functions, which describes processes in porous media. In paper [8] the generalization of IPS is formulated: in paper was proved the existence and uniqueness of generalized integral parabolic spline (GIPS). Generalized integral spline describes processes in double layered systems, where there is an intermediate layer between both layers. Quite often the intermediate layer can be thin, and then with the help of conservative averaging method it can be approximated with linear function. In other words: the splines approximating integral values is defined, both- in the form of polynomial as well as rational, which can also describe the border layers. Their advantage is that their definition includes conditions, that they fulfil conservation laws, which are accomplished in partial differential equations. They can be used in homogeneous media as well as in layered media, and in the cases, where in some places there can be leap in solution or its first derivative. But in all cases continuous functions were considered, which on the boundary of two layers continuously moves from one layer to another. But for the concentration fields in the cases of chemical reaction the leap of first kind for function or its derivative is possible. It was made and was reflected the paper [9].Here a new, normalized form of both splines is offered.

We consider the domain $\Omega \subset R^1$. Let it be given a piecewise-continuous, piecewise-smooth function $U(x), \quad x \in [a,b]$. Also let it be given, that in the different inner points $x_i, \quad i = \overline{1,N}$ the function or its first derivative $U'(x)$ of the function can have a finite jump with $\Theta_i, i = \overline{1,N}$ in the case of function $U(x), \quad x \in [a,b]$ and $Q_i, i = \overline{1,N}$ in the case of function first derivative $U'(x)$. In two figures there are shown the geometrical interpretation of the reconstractable piecewise smooth function $U(x)$ and its first derivative $U'(x)$.





Also can be mentioned that in the case, when $\Theta_i = 0,\ i = \overline{1, N}$ we obtain continuous function and Likely when $Q_i = 0,\ i = \overline{1, N}$, then we obtain that the first derivatives is smooth. If $\Theta_i = Q_i = 0,\ i = \overline{1, N}$ can use IPS proposed in [5] - [8].

Let it be given a piecewise-continuous and piecewise-smooth function $U(x), \quad x \in [a, b]$. Also let it be given, that in the different inner points $x_j, \quad i = \overline{1, N}$ the first derivative $U'(x)$ of the function has a finite jump:

$$k_{i-1} U'(x_i - 0) + \frac{Q_i}{2} = k_i U'(x_i + 0) - \frac{Q_i}{2}.$$

(1.1)

Here coefficients $k_i > 0, \quad i = \overline{0, N}$ are known for all $x \in (x_i, x_{i+1}), \quad x_0 = a, \quad x_{N+1} = b$. Since the function $U(x)$ is continuous on the closed interval $[a, b]$, we additionally have following dis continuity equalities in these points:

$$U(x_i - 0) + \frac{\Theta_i}{2} = U(x_i + 0) - \frac{\Theta_i}{2}, \quad i = \overline{1, N}.$$

(1.2)

Additionally is given the integral averaged values u_i of the function $U(x)$ over the all sub-segments $[x_i . x_{i+1}]$:

$$u_i = H_i^{-1} \int_{x_i}^{x_{i+1}} U(x) dx, \quad H_i = x_{i+1} - x_i, \quad i = \overline{0, N}$$

(1.3)

The interpolation problem consists in approximate reconstruction of the function $U(x)$. This procedure is based on conditions (1)-(3) and following general boundary conditions (BC) on end points $x = a$ and $x = b$:

$$-\nu_0 k_0 U'(a) + \lambda_0 U(a) = \Phi_0$$

(1.4)

$$\nu_1 k_N U'(b) + \lambda_1 U(b) = \Phi_1,$$

$$\nu_l = 0{,}1, \quad \nu_l + \lambda_l > 0, \quad l = 0{,}1.$$

(1.5)

This interpolation problem can be solved by polynomial spline IPS, as in the proposed form it fulfils integral averaged values (3):

$$S(x) = u_i + m_i (x - \bar{x}_i) + e_i \left[\frac{(x - \bar{x}_i)^2}{k_i H_i} - \frac{G_i}{12} \right],$$

$$\bar{x}_i = \frac{x_i + x_{i+1}}{2}, \quad G_i = \frac{H_i}{k_i} > 0$$

(1.6)

For the finding of $2(N+1)$ free coefficients, we have the same number of equations (1), (2), (4) and (5). We will use the same methodology for finding spline with jump coefficients as it was shown in [5] - [7]. Here is proved, that coefficient m_i can be represented through coefficients e_i in two forms:

a) (forward form) for $i = \overline{0, N-1}$

$$3m_i k_i (G_i + G_{i+1}) + e_i (G_i + 3G_{i+1}) +$$
$$+ 2e_{i+1} G_{i+1} + 3Q_{i+1} G_{i+1} + 6\Theta_{i+1} = 6(u_{i+1} - u_i)$$
$$(1.7_a)$$

b) (backward form) for $i = \overline{1, N}$

$$3k_i m_i (G_i + G_{i-1}) - e_i (G_i + 3G_{i-1}) -$$
$$- 2e_{i-1} G_{i-1} - 3Q_i G_{i-1} + 6\Theta_i = 6(u_i - u_{i-1})$$
$$(1.7_b)$$

The elimination of the coefficients m_i from these expressions gives for $i = \overline{1, N - 1}$ us following system regarding the coefficients e_i:

$$a_i e_{i-1} + c_i e_i + b_i e_{i+1} =$$
$$= f_i^- u_{i-1} - f_i u_i + f_i^+ u_{i+1} +$$
$$+ 3Q_{i+1} a_i + 3Q_i b_i + 2\Theta_{i+1} f_i^+ - 2\Theta_i f_i^-,$$
$$i = \overline{1, N - 1}$$
$$(1.8)$$

Here $f_i = f_1^- + f_i^+$ and

$$a_i = \frac{G_{i-1}}{G_i + G_{i-1}}, \quad b_i = \frac{G_{i+1}}{G_i + G_{i+1}},$$
$$f_i^- = \frac{3}{G_i + G_{i-1}}, \quad f_i^+ = \frac{3}{G_i + G_{i+1}}$$
$$(1.9)$$

For the transformation of the BC (4), (5) some additional notations must be used and two different cases are distinguished:

1) $\lambda_0 \neq 0$ (and $\lambda_1 \neq 0$). Then

$$G_{-1} = \frac{2v_0}{\lambda_0}, \quad G_{N+1} = \frac{2v_1}{\lambda_1},$$
$$u_{-1} = \frac{\Phi_0}{\lambda_0}, \quad u_{N+1} = \frac{\Phi_1}{\lambda_1}$$
$$(1.10_0)$$

2) $\lambda_0 = 0$ (and $\lambda_1 = 0$). Then

$$G_{-1} = 2v_0 - G_0, \quad G_{N+1} = 2v_1 - G_N,$$
$$u_{-1} = \Phi_0 + u_0, \quad u_{N+1} = \Phi_1 + u_N$$
$$(1.10_1)$$

Following methodology [7], [8] we have obtained boundary conditions from equations (4), (5) for the first and last equations:

$$(1+a_0+b_0)e_0+b_0e_1=$$
$$=f_0^-u_{-1}-f_0u_0+f_0^+u_1+$$
$$+3Q_1a_0+3Q_0b_0+2\Theta_1f_0^+-2\Theta_0f_0^-,$$
$$a_Ne_{N-1}+(1+a_N+b_N)e_N=$$
$$=f_N^-u_{N-1}-f_Nu_N+f_N^+u_{N+1}+$$
$$+3Q_{N+1}a_N+3Q_Nb_N+2\Theta_{N+1}f_N^+-2\Theta_Nf_N^-.$$
(1.11)

In the case $\lambda_0=0$ $(\lambda_1=0)$ we have special formulas for coefficients a_0 and b_N: $a_0=b_N=1$. In the papers [7], [8] was proposed the representation for coefficients e_i of IPS through all averaged integral values. This representation shows in explicit form also the influence of the BC type and its right hand side on the spline:

$$e_i=\gamma_i^{(0)}f_0^-u_{-1}+\gamma_i^{(1)}f_N^+u_{N+1}+\sum_{j=0}^{N}\beta_{i,j}u_j,$$
$$i=\overline{0,N}$$
(1.12)

The coefficients in the representation (12) are determinate from three systems of linear algebraic equations.

a) The system for $\gamma_i^{(0)}$ (shows the impact of BC (4)):

$$\begin{cases}(1+a_0+b_0)\gamma_0^{(0)}+b_0\gamma_1^{(0)}=1\\ a_i\gamma_{i-1}^{(0)}+(1+a_i+b_i)\gamma_i^{(0)}+b_i\gamma_{i+1}^{(0)}=\\ =0,\quad i=\overline{1,N-1}\\ a_N\gamma_{N-1}^{(0)}+(1+a_N+b_N)\gamma_N^{(0)}=0\end{cases}$$
(1.13_0)

b) The system for $\gamma_i^{(1)}$ (shows the impact of BC (5)):

$$\begin{cases}(1+a_0+b_0)\gamma_0^{(1)}+b_0\gamma_1^{(1)}=0\\ a_i\gamma_{i-1}^{(1)}+(1+a_i+b_i)\gamma_i^{(1)}+b_i\gamma_{i+1}^{(1)}=\\ =0,\quad i=\overline{1,N-1}\\ a_N\gamma_{N-1}^{(1)}+(1+a_N+b_N)\gamma_N^{(1)}=1\end{cases}$$
(1.13_1)

c) The $N+1$ systems ($j=\overline{0,N}$) for β_{ij}:

$$\begin{cases}(1+a_0+b_0)\beta_{0,j}+b_0\beta_{1,j}=\\ =-f_0\delta_{0,j}+f_0^+\delta_{1,j}\\ a_i\beta_{i-1,j}+(1+a_i+b_i)\beta_{i,j}+b_i\beta_{i+1,j}=\\ =f_j^-\delta_{i-1,j}-f_j\delta_{i,j}+f_j^+\delta_{i+1,j},\ i=\overline{1,N-1}\\ a_N\beta_{N-1,j}+(1+a_N+b_N)\beta_{N,j}=\\ =f_N^-\delta_{N-1,j}-f_N\delta_{N,j}\end{cases}$$
(1.14)

where $\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$.

2. Multilayered systems

For the multilayered systems the thickness of the layer is small comparing to system longitude, therefore the conservative averaging method [10] can be applied towards the thickness of layers. As a result a following type of mathematical model is obtained:

$$\frac{\partial T}{\partial t} = a_1^2(x,y)\frac{\partial^2 T}{\partial x^2} + a_2^2(x,y)\frac{\partial^2 T}{\partial y^2},\ x \neq x^{(k)}, y, t > 0,$$

$$\frac{\partial T}{\partial t} = a_k^2(x,y)\frac{\partial^2 T}{\partial y^2} + \frac{\beta_k^+}{2}\frac{\partial T}{\partial x}\bigg|_{x=x^{(k)}+0} - \frac{\beta_k^-}{2}\frac{\partial T}{\partial x}\bigg|_{x=x^{(k)}-0} - w_k\frac{\partial T}{\partial y} + \alpha_k(\Theta_k - T),$$

$$x = x^{(k)}, y, t > 0, \tag{2.1}$$

$$\frac{\partial \Theta_k}{\partial t} = b_k^2(x,y)\frac{\partial^2 \Theta}{\partial y^2} - w_k^0\frac{\partial \Theta_k}{\partial x} + \alpha_k^0(T - \Theta_k),\ \Theta_k\big|_{y=0} = \Theta_k^1(t),$$

$$T\big|_{t=0} = T^0(x,y),\ \Theta_k\big|_{t=0} = \Theta_k^0(y),\ T\big|_{x=x_1^{(k)}, y=0} = T_k^1(t), T\big|_{x\neq x_1^{(k)}, y=0} = T^1(x,t).$$

We consider the mathematical model where separate temperatures are averaged by the thickness of layers and the special type of parabolic splines [5] – [9] are used for the initial mathematical problem simplification. Now, by easiest one-dimensional multi-layer temperature conduction model, we will show how IPS can be used to simplify the initial problem. We consider following heat conduction model with piecewise-constant coefficients:

$$c_i\frac{\partial U_i}{\partial t} = \frac{\partial}{\partial x}(k_i\frac{\partial^2 U_i}{\partial x}) + F_i(x,t), x_i < x < x_{i+1}, i = \overline{0,N}, x_0 = a, x_{N+1} = b, t \in (0,T],$$
(2.2)

$$U_{i-1}(x_i - 0) = U_i(x_i + 0), k_{i-1}\frac{\partial U_{i-1}(x_i - 0)}{\partial x} = k_i\frac{\partial U_i(x_i + 0)}{\partial x},$$
(2.3)

$$-\nu_0 k_0\frac{\partial U_0(a)}{\partial x} + \lambda_0 U_0(a) = \Phi_0(t), \nu_1 k_N\frac{\partial U_N(b)}{\partial x} + \lambda_N U_N(b) = \Phi_1(t),$$
(2.4)

$$U_i(x,0) = U_i^0(x).$$
(2.5)

We introduce in conformity with the method of conservative averaging the integral averaged values $u_i(t)$:

$$u_i(t) = \frac{1}{H_i}\int_{x_i}^{x_{i+1}} U_i(x,t)dx, H_i = x_{i+1} - x_i, i = 0,\dots,N.$$
(2.6)

The integration of main equation (6) gives exact consequences:

$$c_i \frac{du_i}{dt} = \frac{k_i}{H_i} \frac{\partial U_i}{\partial x}\Bigg|_{x=x_i}^{x=x_{i+1}} + f_i(t), i = \overline{0, N}, \quad f_i(t) = \frac{1}{H_i} \int\limits_{x_i}^{x_{i+1}} F_i(x,t)dx.$$

It remains to replace the first term in the right hand side of the equation (10) by the first derivative difference
of IPS:

$$\frac{k_i}{H_i} \frac{\partial U_i}{\partial x}\Bigg|_{x=x_i}^{x=x_{i+1}} \approx \frac{k_i}{H_i} \frac{dS}{dx}\Bigg|_{x=x_i}^{x=x_{i+1}} = \frac{2e_i}{H_i}$$

As the last step, we use the representation for the spline coefficient e_i for $Q_i = \Theta_i = 0, i = \overline{0, N}$:

$$e_i = \gamma_i^{(0)} f_0^- u_{-1} + \gamma_i^{(1)} f_N^+ u_{N+1} + \sum_{j=0}^{N} \beta_{ij} u_j, \ i = \overline{0, N}.$$

This finally gives:

$$c_i \frac{du_i}{dt} = \frac{2}{H_i} \left[\sum_{j=0}^{N} \beta_{ij} u_j + \gamma_i^{(0)} f_0^- u_{-1} + \gamma_i^{(1)} f_N^+ u_{N+1} \right] + f_i(t), i = \overline{0, N}, t \in (0, T].$$

(2.7)

To this system of ordinary differential equations (ODE) averaged over sub-segments $[x_i, x_{i+1}]$, initial conditions (9) must be added (the conjugations conditions and BC were already used by construction of IPS):

$$u_i(0) = u_i^0 := \frac{1}{H_i} \int\limits_{x_i}^{x_{i+1}} U_i^0(x)dx .$$

(2.8)

So, we have reduced the one-dimensional heat conduction problem with discontinuous coefficients for PDE to the system of ODE with continuous coefficients. After solving problem (12), (13), we can approximately reconstruct the solution of original problem by IPS (2). The approximation error is known.

Generalized Integral Parabolic Spline. Let it again be given a continuous, piecewise-smooth function $U(x), x \in [a, b]$, for which the first derivative $U'(x)$ has first kind discontinuities in $2N$ different inner points $x_i, x_{i-1/2}, \ i = 1, \ldots, N$:

$$k_{i-1} U'(x_{i-1/2} - 0) = k_{i-1/2} U'(x_{i-1/2} + 0), k_{i-1/2} U'(x_i - 0) = k_i U'(x_i + 0).$$

The continuity property of the function $U(x)$ gives following $2N$ equalities:

$$U(x_{i-1/2} - 0) = U(x_{i-1/2} + 0), U(x_i - 0) = U(x_i + 0).$$

Here the function (coefficients) $k(x)$ is piecewise-constant function:

$$k(x) = \begin{cases} k_{i-1/2}, if \ x \in (x_{i-1/2}, x_i), \\ k_i, \quad if \ x \in (x_i, x_{i+1/2}) \end{cases}$$

with property $k_{i-1/2} \square k_{i-1}, \ k_{i-1/2} \square k_i$. Here must be mentioned that the $x_{N+1/2} = x_{N+1} = b$, it means the function $U(x)$ can neither end nor begin with a linear part of the segment $[a, b]$.

Additionally are given $N + 1$ values u_i of the function $U(x)$ over sub-segments $[x_i, x_{i+1/2}]$:

$$\tilde{u}_i = \frac{1}{\tilde{H}_i} \int\limits_{x_i}^{x_{i+1/2}} U(x)dx, \ \tilde{H}_i = x_{i+1/2} - x_i, i = 0, N.$$

(2.9)

Further, it is known, that on the sub-segments $[x_{i+1/2}, x_{i+1}], i = \overline{0, N-1}$ the function can be approximated by linear function. Finally, the BC must be fulfilled.
In our paper [7], [8] it was shown that there exists exactly one spline fulfilling all mentioned conditions. We will seek this spline in the form:

$$\tilde{S}(x) = \begin{cases} \tilde{u}_i + \tilde{m}_i (x - \tilde{x}_i) + \tilde{e}_i \left[\dfrac{(x-\tilde{x}_i)^2}{k_i \tilde{H}_i} - \dfrac{\tilde{G}_i}{12} \right], \ x \in [x_i, x_{i+1/2}], \tilde{x}_i = \dfrac{x_i + x_{i+1/2}}{2}, i = \overline{0, N}; \\ u_{i-1/2} + m_{i-1/2}(x - \overline{x}_{i-1/2}), x \in [x_{i-1/2}, x_i], \ \overline{x}_{i-1/2} = \dfrac{x_{i-1/2} + x_i}{2}, i = \overline{1, N}. \end{cases}$$

For explicitness of presentation, we consider following one-dimensional heat conduction (or diffusion) model with piecewise-constant coefficients:

$$\tilde{c}_i \frac{\partial \tilde{U}_i}{\partial t} = \frac{\partial}{\partial x}(\tilde{k}_i \frac{\partial^2 \tilde{U}_i}{\partial x}) - q_i(t)\tilde{U}_i + F_i(x,t),$$

$$x_i < x < x_{i+1/2}, i = \overline{0, N}, \ x_0 = a, \ x_{N+1/2} = x_{N+1} = b, \ t \in (0, T]. \tag{2.10}$$

$$c_{i-1/2} \frac{\partial U_{i-1/2}}{\partial t} = \frac{\partial}{\partial x}(k_{i-1/2} \frac{\partial^2 U_{i-1/2}}{\partial x}) - q_{i-1/2} U_{i-1/2}$$

$$+ F_{i-1/2}(x,t), \quad x_{i-1/2} < x < x_i, \ i = \overline{1, N}. \tag{2.11}$$

To the differential equations (33) for separate elementary bars and the equations (34) for interlayers we must add the conjugations conditions in the connection points $x_i, x_{i-1/2}$:

$$\tilde{U}_{i-1}(x_{i-1/2} - 0) = U_{i-1/2}(x_{i-1/2} + 0),$$

$$U_{i-1/2}(x_i - 0) = \tilde{U}_i(x_i + 0), i = \overline{1, N}, \tag{2.12}$$

$$\tilde{k}_{i-1} \frac{\partial \tilde{U}_{i-1}(x_{i-1/2} - 0)}{\partial x} = k_{i-1/2} \frac{\partial U_{i-1/2}(x_{i-1/2} + 0)}{\partial x},$$

$$k_{i-1/2} \frac{\partial U_{i-1/2}(x_i - 0)}{\partial x} = \tilde{k}_i \frac{\partial \tilde{U}_i(x_i + 0)}{\partial x}. \tag{2.13}$$

Finally, to choose unique solution, we must add BC for the first and last elementary bars and initially conditions for all elements of the multilayer bar:

$$-\nu_0 k_0 \frac{\partial U_0(a)}{\partial x} + \lambda_0 U_0(a) = \Phi_0(t),$$

$$\nu_1 k_N \frac{\partial U_N(b)}{\partial x} + \lambda_1 U_N(b) = \Phi_1(t), \tag{2.14}$$

$$\tilde{U}_i(x,0) = \tilde{U}_i^0(x), U_{i-1/2}(x,0) = U_{i-1/2}^0(x). \tag{2.15}$$

As one can see, the coefficients have discontinuities in $2N$ different inner points $x_i, x_{i-1/2}, \ i = 1, \ldots, N$.

In technical (or natural) systems, often it may happen that interlayers' physical and geometrical properties fulfil following inequalities:

$k_{i-1/2} \ll \min\left(k_{i-1}, k_i\right), i = \overline{1, N},$

$x_i - x_{i-1/2} = H_{i-1/2} < \min\left(\tilde{H}_{i-1}, \tilde{H}_i\right).$

The mathematical modelling in such cases restricts oneself to assumption that interlayers' solution is linear regarding argument x. In this and in next sub-sections we will specifically investigate this case.

The solution of problem (33)-(38) is continuous in closed rectangle $\{x \in [a,b]\} \otimes \{t \in [0,T]\}$ function

$$U(x,t) = \begin{cases} \tilde{U}_i(x,t), if\ x \in (x_i, x_{i+1/2}), i = \overline{0, N}, \\ U_{i-1/2}(x,t), if\ x \in (x_{i-1/2}, x_i), i = \overline{1, N}. \end{cases}$$

The integral averaged values $\tilde{u}_i(t)$ are introduced in consistency with GIPS property:

$$\tilde{u}_i(t) = \frac{1}{\tilde{H}_i} \int_{x_i}^{x_{i+1/2}} \tilde{U}_i(x,t)dx, \tilde{H}_i = x_{i+1/2} - x_i, i = \overline{0, N}.$$

We integrate the main differential equation (10):

$$\tilde{c}_i \frac{d\tilde{u}_i}{dt} = \frac{\tilde{k}_i}{\tilde{H}_i} \frac{\partial \tilde{U}_i}{\partial x}\Bigg|_{x=x_i}^{x=x_{i+1/2}} - q_i(t)\tilde{u}_i + f_i(t),\ f_i(t) = \frac{1}{\tilde{H}_i} \int_{x_i}^{x_{i+1/2}} F_i(x,t)dx.$$

(2.16)

We underline that the equality (16) is exact, not approximate consequence of PDE (10). Now the approximation will be made by replacing the first term in the right hand side with difference of splines' first derivatives:

$$\frac{\tilde{k}_i}{\tilde{H}_i} \frac{\partial \tilde{U}_i}{\partial x}\Bigg|_{x=x_i}^{x=x_{i+1/2}} \approx \frac{\tilde{k}_i}{\tilde{H}_i} \frac{d\tilde{S}}{dx}\Bigg|_{x=x_i}^{x=x_{i+1/2}} = \frac{2\tilde{e}_i}{\tilde{H}_i}.$$

(2.17)

The substitution of the representation in formula (17) gives approximate equality:

$$\frac{\tilde{k}_i}{\tilde{H}_i} \frac{\partial \tilde{U}_i}{\partial x}\Bigg|_{x=x_i}^{x=x_{i+1/2}} \approx \frac{2}{\tilde{H}_i}\left[\sum_{j=0}^{N} \tilde{\beta}_{ij}\tilde{u}_j + \tilde{\gamma}_i^{(0)} \tilde{f}_0^- \tilde{u}_{-1} + \tilde{\gamma}_i^{(1)} \tilde{f}_N^+ \tilde{u}_{N+1}\right].$$

(2.18)

Now we substitute the right hand side of formula (17) in the equation (16) and we obtain the final form of the averaged differential equation for the elementary bars together with averaged initial condition:

$$\tilde{c}_i \frac{d\tilde{u}_i}{dt} = \frac{\tilde{k}_i}{\tilde{H}_i}\left[\sum_{j=0}^{N} \tilde{\beta}_{ij}\tilde{u}_j + \tilde{\gamma}_i^{(0)} \tilde{f}_0^- \tilde{u}_{-1} + \tilde{\gamma}_i^{(1)} \tilde{f}_N^+ \tilde{u}_{N+1}\right] - q_i(t)\tilde{u}_i + f_i(t),$$

$$\tilde{u}_i(0) = \frac{1}{\tilde{H}_i} \int_{x_i}^{x_{i+1/2}} \tilde{U}_i^0(x)dx.$$

(2.19)

The solution (analytically or numerically) of this Cauchy problem for the system of ODE (28) gives us the functions $\tilde{u}_i(t), i = \overline{0, N}$. This allows us calculate from representation (16) all coefficients $\tilde{e}_i, \tilde{m}_i$ of GIPS and thus reconstruct solution $\tilde{U}_i(x,t), i = \overline{0, N}$ for all elementary bars.

The multi-layered systems for discontinuous piecewise functions was been developed Andris Buikis, Margarita Buike, together with Solvita Kostjukova, doctoral student. This research was also noticed by other scientists like Prof. Siavash Sohrab from U.S. because the spline was suitable for the shockwave analysis.

The research on the mathematical models of wood fibre and plywood has been carried out, where the mentioned integral spline has been applied. The existing mathematical models have been inspected by Cepitis et al., Buikis et al., Kang et al. and new ones have been developed. Numerical simulations, extensive study of the physical non-linear properties as well as the comparison of the models has been carried out.

Another research activity was related to electric systems like automotive fuses and insulated wires. R. Vilums obtained doctor's degree in 2010 regarding this subject. The mathematical models of heating of fuses developed with the conservative averaging method (CAM) (Vilums (2010), Vilums et al. (2008a and 2008b)) were improved, numerically solved and compared with the results obtained by the standard finite volume approach (Vilums (2011)). Temperature-dependant physical coefficients were used and free heat convection and heat radiation on the surface were considered.

Numerical implementation of the models was carried out in Maple and in an open-source C++ library called OpenFOAM, which was used for the finite volume calculations. Because OpenFOAM has insufficient documentation in general and none in Latvian, several tutorial examples in Latvian have been created which included the implementation of the mathematical models of insulated wire and automotive fuse. These documents and other useful information collected during the research were made available on a web site.

3. Wall with two fins

In this section we present mathematical three dimensional formulation of a transient problem one element with two rectangular fins attached to both sides [13].
We will use following dimensionless arguments, parameters:

$$x = x'/(B+R),\ y = y'/(B+R),\ z = z'/(B+R),\ \ l = L/(B+R),\ l_1 = L_1/(B+R),\ w = W/(B+R),$$

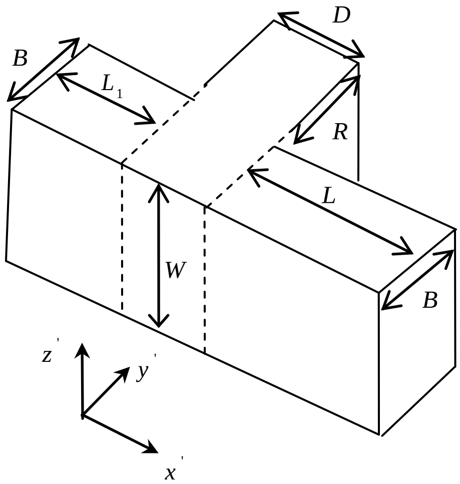
$$b = B/(B+R),\ \delta = D/B+R,\ \ \beta = hk^{-1}(B+R),\ \beta_0 = h_0 k^{-1}(B+R)$$

and dimensionless temperatures:

$$\bar{V}(x,y,z,t) = \frac{\tilde{V}(x,y,z,t) - T_a(t)}{T_b(t) - T_a(t)},\ \bar{V}_0(x,y,z,t) = \frac{\tilde{V}_0(x,y,z,t) - T_a(t)}{T_b(t) - T_a(t)},$$

$$\bar{V}_1(x,y,z,t) = \frac{\tilde{V}_1(x,y,z,t) - T_a(t)}{T_b(t) - T_a(t)},\ \bar{\Theta}(x,y,z,t) = \frac{\tilde{\Theta}(x,y,z,t) - T_a(t)}{T_b(t) - T_a(t)},$$

$$\bar{\Theta}_0(x,y,z,t) = \frac{\tilde{\Theta}_0(x.y,z,t) - T_a(t)}{T_b(t) - T_a(t)}.$$



Heat exchanger consisting of rectangular fins attached on either sides of a plane wall

We have introduced following dimensional thermal and geometrical parameters: k - heat conductivity coefficients for the wall, right fin and left fin, $h(h_0)$ - heat exchange coefficient for the right (left) side, $2B$ – fins width (thickness), L – right fin length, L_1 – left fin length, D - thickness of the wall, W - walls' width (length), $2R$ – distance between two fins (fin spacing). Further, $\tilde{\Theta}_0(x.y,z,t)$ is the surrounding (environment) temperature on the left (hot) side (the heat source side) of the wall, $\tilde{\Theta}(x,y,z,t)$ - the surrounding temperature on the right (cold - the heat sink side) of the wall and the fin. Finally $\tilde{V}_0(x,y,z,t)$, $\tilde{V}(x,y,z,t)$, $\tilde{V}_1(x,y,z,t)$ are the dimensional temperatures in the wall, right fin and left fin where $T_a(T_b)$ are integral averaged environment temperatures over appropriate edges.

The one element of the wall (base) is placed in the domain $\{x\in[0,\delta],\ y\in[0,1],\ z\in[0,w]\}$. The rectangular right fin in dimensionless arguments occupies the domain $\{x\in[\delta,\delta+l],\ y\in[0,b],\ z\in[0,w]\}$. The rectangular left fin in dimensionless arguments occupies the domain $\{x\in[-l_1,0],\ y\in[0,b],\ z\in[0,w]\}$. We describe the temperature field by functions $\bar{V}(x,y,z,t)$, $\bar{V}_0(x,y,z,t)$, $\bar{V}_1(x,y,z,t)$ in the wall and fins:

$$\frac{\partial^2\bar{V}_0}{\partial x^2}+\frac{\partial^2\bar{V}_0}{\partial y^2}+\frac{\partial^2\bar{V}_0}{\partial z^2}=\frac{1}{a^2}\frac{\partial\bar{V}_0}{\partial t}, \tag{3.1}$$

$$\frac{\partial^2\bar{V}}{\partial x^2}+\frac{\partial^2\bar{V}}{\partial y^2}+\frac{\partial^2\bar{V}}{\partial z^2}=\frac{1}{a^2}\frac{\partial\bar{V}}{\partial t}, \tag{3.2}$$

$$\frac{\partial^2\bar{V}_1}{\partial x^2}+\frac{\partial^2\bar{V}_1}{\partial y^2}+\frac{\partial^2\bar{V}_1}{\partial z^2}=\frac{1}{a^2}\frac{\partial\bar{V}_1}{\partial t}. \tag{3.3}$$

We must add initial conditions for the heat equations (1)–(3):

$$\bar{V}_0\Big|_{t=0}=V_0^0(x,y,z), \tag{3.4}$$

$$\bar{V}\Big|_{t=0} = V^{0}(x, y, z),$$

(3.5)

$$\bar{V}_1\Big|_{t=0} = V_1^{0}(x, y, z).$$

(3.6)

We assume heat fluxes from the flank surfaces (edges) and from the top and the bottom edges:

$$\frac{\partial \bar{V}_0}{\partial z}\Big|_{z=0} = Q_{0,2}(x, y, t), \quad \frac{\partial \bar{V}_0}{\partial z}\Big|_{z=w} = Q_{0,3}(x, y, t),$$

(3.7)

$$\frac{\partial \bar{V}}{\partial z}\Big|_{z=0} = Q_2(x, y, t), \quad \frac{\partial \bar{V}}{\partial z}\Big|_{z=w} = Q_3(x, y, t),$$

(3.8)

$$\frac{\partial \bar{V}_1}{\partial z}\Big|_{z=0} = Q_{1,2}(x, y, t), \quad \frac{\partial \bar{V}_1}{\partial z}\Big|_{z=w} = Q_{1,3}(x, y, t).$$

(3.9)

In the case of steady state problem all above mentioned functions are time-independent. Three dimensional formulation of a steady state problem can be obtained in the similar way. Instead of (1) –(3) we have:

$$\frac{\partial^2 \bar{V}_0}{\partial x^2} + \frac{\partial^2 \bar{V}_0}{\partial y^2} + \frac{\partial^2 \bar{V}_0}{\partial z^2} = 0, \quad \frac{\partial^2 \bar{V}}{\partial x^2} + \frac{\partial^2 \bar{V}}{\partial y^2} + \frac{\partial^2 \bar{V}}{\partial z^2} = 0, \quad \frac{\partial^2 \bar{V}_1}{\partial x^2} + \frac{\partial^2 \bar{V}_1}{\partial y^2} + \frac{\partial^2 \bar{V}_1}{\partial z^2} = 0.$$

Initial conditions (4)-(6) are not needed. Conditions (7)-(9) are in the form:

$$\frac{\partial \bar{V}_0}{\partial z}\Big|_{z=0} = Q_{0,2}(x, y), \quad \frac{\partial \bar{V}_0}{\partial z}\Big|_{z=w} = Q_{0,3}(x, y), \quad \frac{\partial \bar{V}}{\partial z}\Big|_{z=0} = Q_2(x, y), \quad \frac{\partial \bar{V}}{\partial z}\Big|_{z=w} = Q_3(x, y),$$

$$\frac{\partial \bar{V}_1}{\partial z}\Big|_{z=0} = Q_{1,2}(x, y), \quad \frac{\partial \bar{V}_1}{\partial z}\Big|_{z=w} = Q_{1,3}(x, y).$$

Such type of boundary conditions (BC) (7) – (9) allows us to make the exact reducing of this three dimensional problem to the two dimensional problem by conservative averaging method [10]. Let us introduce following integral averaged values:

$$V_0(x, y, t) = w^{-1} \cdot \int_0^w \bar{V}_0(x, y, z, t)dz,$$

(3.10)

$$\vartheta_0(x, y, t) = w^{-1} \cdot \int_0^w \Theta_0(x, y, z, t)dz,$$

(3.11)

$$V(x, y, t) = w^{-1} \cdot \int_0^w \bar{V}(x, y, z, t)dz,$$

(3.12)

$$\vartheta(x, y, t) = w^{-1} \cdot \int_0^w \Theta(x, y, z, t)dz,$$

(3.13)

$$\bar{V}_1(x,y,t) = w^{-1}\cdot\int_0^w V_1(x,y,z,t)dz.$$

(3.14)

Realizing the integration of main equations (1) – (3) by usage of the BC (7) – (12) we obtain:

$$\frac{\partial^2 V_0}{\partial x^2}+\frac{\partial^2 V_0}{\partial y^2}+Q_0(x,y,t)=\frac{1}{a^2}\frac{\partial V_0}{\partial t},$$

(3.15)

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+Q(x,y,t)=\frac{1}{a^2}\frac{\partial V}{\partial t},$$

(3.16)

$$\frac{\partial^2 V_1}{\partial x^2}+\frac{\partial^2 V_1}{\partial y^2}+Q_1(x,y,t)=\frac{1}{a^2}\frac{\partial V_1}{\partial t}.$$

(3.17)

Here

$$Q_0(x,y,t)=w^{-1}\left(Q_{0,3}(x,y,t)-Q_{0,2}(x,y,t)\right),\ Q(x,y,t)=w^{-1}\left(Q_3(x,y,t)-Q_2(x,y,t)\right),$$

$$Q_1(x,y,t)=w^{-1}\left(Q_{1,3}(x,y,t)-Q_{1,2}(x,y,t)\right).$$

We add to the main partial differential equations (15) – (17) needed BC as follows:

$$\left.\left(\frac{\partial V_0}{\partial x}+\beta_0\left[\vartheta_0(x,y,t)-V_0\right]\right)\right|_{x=0}=0,\ y\in(b,1),$$

(3.18)

$$\left.\left(\frac{\partial V_0}{\partial x}+\beta\left[V_0-\vartheta(x,y,t)\right]\right)\right|_{x=\delta}=0,\ y\in(b,1),$$

(3.19)

$$\left.\frac{\partial V_0}{\partial y}\right|_{y=0}=Q_{0,0}(x,t),\ x\in(0,\delta),$$

(3.20)

$$\left.\frac{\partial V_0}{\partial y}\right|_{y=1}=Q_{0,1}(x,t),\ x\in(0,\delta).$$

(3.21)

We assume them as ideal thermal contact between wall and fins - there is no contact resistance:

$$V_0\big|_{x=\delta-0}=V\big|_{x=\delta+0},$$

(3.22)

$$\left.\frac{\partial V_0}{\partial x}\right|_{x=\delta-0}=\left.\frac{\partial V}{\partial x}\right|_{x=\delta+0},$$

(3.23)

$$V_1\big|_{x=0-0}=V_0\big|_{x=0+0},$$

(3.24)

$$\left.\frac{\partial V_1}{\partial x}\right|_{x=0-0}=\left.\frac{\partial V_0}{\partial x}\right|_{x=0+0}.$$

(3.25)

We have following BC for the right fin:

$$\left.\left(\frac{\partial V}{\partial x}+\beta\left[V-\vartheta(x,y,t)\right]\right)\right|_{x=\delta+l}=0,\ y\in(0,b),$$
(3.26)
$$\left.\left(\frac{\partial V}{\partial y}+\beta\left[V-\vartheta(x,y,t)\right]\right)\right|_{y=b}=0,\ x\in(\delta,\delta+l),$$
(3.27)
$$\left.\frac{\partial V}{\partial y}\right|_{y=0}=Q_0(x,t),\ x\in(\delta,\delta+l).$$
(3.28)
We have following BC for the left fin:

$$\left.\left(\frac{\partial V_1}{\partial x}+\beta_0\left[\vartheta_0(x,y,t)-V_1\right]\right)\right|_{x=-l_1}=0,\ y\in(0,b),$$
(3.29)
$$\left.\left(\frac{\partial V_1}{\partial x}+\beta_0\left[V_1-\vartheta_0(x,y,t)\right]\right)\right|_{y=b}=0,\ x\in(-l_1,0),$$
(3.30)
$$\left.\frac{\partial V_1}{\partial y}\right|_{y=0}=Q_1(x,t),\ x\in(-l_1,0).$$
(3.31)
Finally, we introduce integral averaged values as (10) – (14) and add initial conditions for the heat equations (15) – (17):
$$V_0\big|_{t=0}=V_0^0(x,y),$$
(3.32)
$$V\big|_{t=0}=V^0(x,y),$$
(3.33)
$$V_1\big|_{t=0}=V_1^0(x,y).$$
(3.34)
In the similar way three dimensional steady state problem can be reduced to the two dimensional problem. Instead of (15)–(17) we have:
$$\frac{\partial^2 V_0}{\partial x^2}+\frac{\partial^2 V_0}{\partial y^2}+Q_0(x,y)=0,\quad \frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+Q(x,y)=0,\quad \frac{\partial^2 V_1}{\partial x^2}+\frac{\partial^2 V_1}{\partial y^2}+Q_1(x,y)=0.$$
BC are still in the form (18)-(21), (26)-(31), conjugation conditions are in the form (22)-(25) for time-independent functions $V_0(x,y), V(x,y), V_1(x,y), \vartheta_0(x,y), \vartheta(x,y)$. Initial conditions (32)-(34) are not needed for steady state problem.
This section represents solution for the 2D case of periodical system with constant dimensionless environmental temperatures $\vartheta_0=1(\Theta_0=T_b)$ and $\vartheta=0(\Theta=T_a)$. We consider $U(x,y)$ is a temperature of the right fin, $U_0(x,y)$ is a temperature of the wall and $U_1(x,y)$ is a temperature of the left fin. Thus, the main equations are:
$$\frac{\partial^2 U_0}{\partial x^2}+\frac{\partial^2 U_0}{\partial y^2}=0,$$
(3.35)

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0,$$
(3.36)
$$\frac{\partial^2 U_1}{\partial x^2} + \frac{\partial^2 U_1}{\partial y^2} = 0.$$
(3.37)
The BC (20), (21), (28), (31) are assumed to be homogeneous:
$$\left.\frac{\partial U_0}{\partial y}\right|_{y=0} = \left.\frac{\partial U_0}{\partial y}\right|_{y=1} = \left.\frac{\partial U}{\partial y}\right|_{y=0} = \left.\frac{\partial U_1}{\partial y}\right|_{y=0} = 0.$$
Instead of BC (18), (19), (26), (27), (29) and (30) we have:
$$\left.\frac{\partial U_0}{\partial x} + \beta_0[1 - U_0]\right|_{x=0} = 0,\ y \in (b, 1),$$
(3.38)
$$\left.\left(\frac{\partial U_0}{\partial x} + \beta U_0\right)\right|_{x=\delta} = 0,\ y \in (b, 1),$$
(3.39)
$$\left.\left(\frac{\partial U}{\partial x} + \beta U\right)\right|_{x=\delta+l} = 0,\ y \in [0, b],$$
(3.40)
$$\left.\left\{\frac{\partial U}{\partial y} + \beta U\right\}\right|_{y=b} = 0,\ x \in [\delta, \delta + l],$$
(3.41)
$$\left.\left(\frac{\partial U_1}{\partial x} + \beta_0[1 - U_1]\right)\right|_{x=-l_1} = 0,\ y \in [0, b],$$
(3.42)
$$\left.\left\{\frac{\partial U_1}{\partial y} + \beta_0[U_1 - 1]\right\}\right|_{y=b} = 0,\ x \in [-l_1, 0].$$
(3.43)
The conjugations conditions on the line between the wall and the left fin are still standing in the form (24), (25) for the functions $U_0(x, y)$ and $U_1(x, y)$. The linear combination of the equations (24), (25) together with BC (38) allow us rewrite them as following BC on the left hand side of the wall:
$$\left.\left(\frac{\partial U_0}{\partial x} - \beta_0 U_0\right)\right|_{x=0+0} = \beta_0 F_1(0, y), \qquad (3.44)$$
where
$$F_1(x, y) = \begin{cases} \dfrac{1}{\beta_0}\dfrac{\partial U_1}{\partial x} - U_1, & 0 \le y \le b,\ 0 \le x \le \delta, \\ -1, & b < y \le 1. \end{cases}$$
(3.45)
In the similar way using the linear combination of the equations (22), (23) together with BC (39) we rewrite following BC on the right hand side of the wall:
$$\left.\left(\frac{\partial U_0}{\partial x} + \beta U_0\right)\right|_{x=\delta-0} = \beta F_0(\delta, y),$$
(3.46)
where

$$F_0(x,y)=\begin{cases}\left(\dfrac{1}{\beta}\dfrac{\partial U}{\partial x}+U\right), 0\le y\le b, 0\le x\le\delta,\\ 0, \qquad\qquad , b<y\le 1.\end{cases}$$

(3.47)

On the assumption that the functions $F_1(0,y)$, $F_0(\delta,y)$ are given we can represent solution for the wall in very well known form by the Green function:

$$U_0(x,y)=-\int_0^1 F_1(0,\nu)G_0(x,y,0,\nu)d\nu+\int_0^b F_0(\delta,\nu)G_0(x,y,\delta,\nu)d\nu,$$

(3.48)

where Green function is:

$$G_0(x,y,\varsigma,\nu)=\sum_{m,n=1}^{\infty}\frac{G^x_{0,m}(x,\varsigma)\cdot G^y_{0,n}(y,\nu)}{\left[(\pi\cdot n)^2+\mu_m^2\right]},$$

$$G^x_{0,m}(x,\varsigma)=\frac{\varphi_{0,m}(x)\varphi_{0,m}(\varsigma)}{\left\|\varphi_{0,m}\right\|^2},\ G^y_{0,n}(y,\nu)=\cos\left[n\pi(y+\nu)\right]+\cos\left[n\pi(y-\nu)\right],$$

$$\varphi_{0,,m}(x)=\cos(\mu_m x)+\frac{\beta_0}{\mu_m}\sin(\mu_m x),\ \left\|\varphi_{0,m}\right\|^2=\frac{\beta_0}{2\mu_m^2}+\frac{\beta}{2\mu_m^2}\frac{\mu_m^2+(\beta_0)^2}{\mu_m^2+(\beta)^2}+\frac{\delta}{2}\left(1+\frac{\beta_0^2}{\mu_m^2}\right).$$

Here μ_m are the positive roots of the transcendental equation:

$$\tan(\mu_m\delta)=\frac{\mu_m(\beta_0+\beta)}{\mu_m^2-\beta_0\beta}.$$

Unfortunately the representation (48) is unusable as solution for the wall because of unknown functions $F_1(0,y)$, $F_0(\delta,y)$ i.e. temperature in the fins. That is why we will pay attention to the solution for the fins now. In the same way we can rewrite the conjugations conditions (22), (23) in the form of BC on the left side of the right rectangular fin:

$$\left.\left(\frac{\partial U}{\partial x}-\beta U\right)\right|_{x=\delta+0}=\beta F(\delta,y),$$

(3.49)

where

$$F(x,y)=\left(\frac{1}{\beta}\frac{\partial U_0}{\partial x}-U_0\right), 0\le y\le b, \delta\le x\le\delta+l.$$

(3.50)

Then, similar as for the wall we can represent solution for the right fin in following form:

$$U(x,y)=-\int_0^b F(\delta,\eta)G(x,y,\delta,\eta)d\eta,$$

(3.51)

where Green function is:

$$G(x,y,\xi,\eta)=\sum_{i,j=1}^{G_m^{(x)}}\frac{G_i^{(x)}(x,\xi)\cdot G_j^{(y)}(y,\eta)}{\lambda_i^2+k_j^2},\ G_i^x(x,\xi)=\frac{\phi_i(x)\phi_i(\xi)}{\left\|\phi_i\right\|^2},$$

$$G_j^{(y)}(y,\eta)=\frac{\psi_j(y,\eta)}{2\left\|\psi_j\right\|^2},\ \phi_i(x)=\cos\left[\lambda_i(x-\delta)\right]+\frac{\beta}{\lambda_i}\sin\left[\lambda_i(x-\delta)\right],\ \left\|\phi_i\right\|^2=\frac{\beta}{\lambda_i^2}+\frac{l}{2}\left(1+\frac{\beta^2}{\lambda_i^2}\right),$$

$$\psi_j(y,\eta)=\cos\left[\kappa_j(y+\eta)\right]+\cos\left[\kappa_j(y-\eta)\right],\ \left\|\psi_j\right\|^2=\frac{1}{2}\left(b+\frac{\beta}{k_j^2+\beta^2}\right).$$

Here λ_i, k_j are the positive roots of the transcendental equations:

$$\tan(\lambda_i l) = \frac{2\lambda_i \beta}{\lambda_i^2 - \beta^2}, \ \tan(\kappa_j b) = \frac{\beta}{\kappa_j}.$$

Finally, we rewrite the conjugations conditions in the form of BC on the right side of the left rectangular fin:

$$\left(\frac{\partial U_1}{\partial x} + \beta_0 U_1\right)\Bigg|_{x=0+0} = \beta_0 F_2(0, y),$$

(3.52)

where

$$F_2(x, y) = \frac{1}{\beta_0}\frac{\partial U_0}{\partial x} + U_0, \ 0 \le y \le b.$$

(3.53)

Thus, solution for the left fin we can represent in following form:

$$U_1(x, y) = \beta_0 \int_0^b G_1(x, y, -l_1, \nu) d\nu + \int_0^b F_2(0, \nu) G_1(x, y, 0, \nu) d\nu + \beta_0 \int_{-l_1}^0 G_1(x, y, \xi, b) d\xi,$$

(3.54)

where

$$G_1(x, y, \xi, \eta) = \sum_{i,j=1}^{G_m^{(x)}} \frac{G_{1,i}^{(x)}(x,\xi)\cdot G_{1,j}^{(y)}(y,\eta)}{\mu_i^2 + \lambda_j^2}, \ G_{1,i}^{x}(x,\xi) = \frac{\varphi_{1,i}(x)\varphi_{1,i}(\xi)}{\left\|\varphi_{1,i}\right\|^2}, \ G_{1,j}^{(y)}(y,\eta) = \frac{\psi_{1,j}(y,\eta)}{2\left\|\psi_{1,j}\right\|^2},$$

$$\varphi_{1,i}(x) = \cos[\mu_i(x + l_1)] + \frac{\beta_0}{\mu_i}\sin[\mu_i(x + l_1)], \ \left\|\varphi_{1,i}\right\|^2 = \frac{\beta_0}{\mu_i^2} + \frac{l_1}{2}\left(1 + \frac{\beta_0^2}{\mu_i^2}\right),$$

$$\psi_{1,j}(y) = \cos[\lambda_j(y - \eta)] + \cos[\lambda_j(y + \eta)], \ \left\|\psi_{1,j}\right\|^2 = \frac{1}{2}\left(b + \frac{\beta_0}{\lambda_j^2 + \beta_0^2}\right).$$

Here μ_i, λ_j are the positive roots of the transcendental equations:

$$\tan(\mu_i l_1) = \frac{2\mu_i \beta_0}{\mu_i^2 - \beta_0^2}, \ \tan(\lambda_j b) = \frac{\beta_0}{\lambda_j}.$$

Using notation (51) and representation (47) we can easy obtain the following equation:

$$F_0(x, y) = -\int_0^b F(\delta, \eta)\cdot\Gamma(x, y, \delta, \eta) d\eta,$$

(3.55)

Where $\Gamma(x, y, \xi, \eta) = \left(\frac{\partial}{\partial x} + \beta\right) G(x, y, \xi, \eta).$

In the similar way we find equation for $F_1(0, y)$ by using (45) and (54):

$$F_1(0, y) = \beta_0 \int_0^b \Gamma_1(0, y, -l_1, \nu) d\nu - \beta_0 \int_{-l_1}^0 \Gamma_1(0, y, \xi, b) d\xi + \int_0^b F_2(0, \nu)\Gamma_1(0, y, 0, \nu) d\nu,$$

(3.56)

where

$$\Gamma_1(x, y, \varsigma, \nu) = \left(\frac{\partial}{\partial x} - \beta_0\right) G_1(x, y, \varsigma, \nu).$$

Next, we find equation for $F(\delta, y)$ by using (50) and (48):

$$F(\delta,\eta) = \int_0^b F_0(\delta,\nu)\Gamma_0(\delta,\eta,\delta,\nu)d\nu - \int_0^1 F_1(0,\nu)\Gamma_0(\delta,\eta,0,\nu)d\nu. \qquad (3.57)$$

Where $\Gamma_0(x,y,\varsigma,\nu) = \left(\frac{\partial}{\partial x} - \beta\right) G_0(x,y,\varsigma,\nu)$.

Finally, using (53) and (48) we get equation for $F_2(0,y)$:

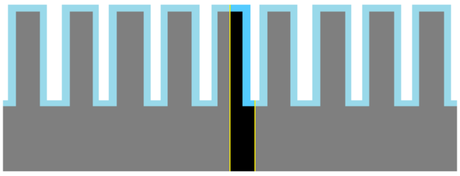
$$F_2(0,y) = -\int_0^1 F_1(0,\nu)\Gamma_2(0,y,0,\nu)d\nu + \int_0^b F_0(\delta,\nu)\Gamma_2(\delta,y,\delta,\nu)d\nu. \qquad (3.58)$$

Here $\Gamma_2(x,y,\varsigma,\nu) = \left(\frac{\partial}{\partial x} + \beta_0\right) G_0(x,y,\varsigma,\nu)$.

When a system of Fredholm integral equations of the second kind (58) – (61) is solved, we obtain the temperatures fields in the wall (48), left fin (54) and right fin (51).

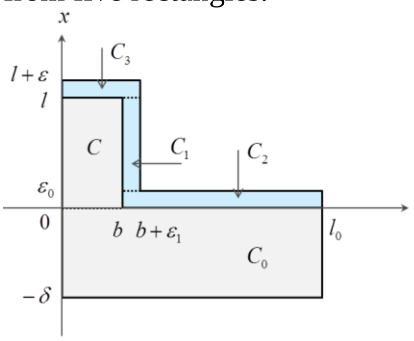
4. System with Double Wall and Double Fins

Since the given system can be divided into several symmetrical parts, we will describe the problem for only one of those L-shaped parts [14].



L-type domain

We are going to represent the original L-type domain as a finite union of canonical subdomains with appropriate conjugation conditions along the lines connecting two neighbour domains. We may therefore suppose that this L-shaped sample is made up from five rectangles.



Definition of geometrical parameters for the sample

Let's assume that $V_i(x,y)$ denotes the temperature in the domain C_i with thermal conductivity k_i, and h_i is heat transfer coefficient. Here $k = k_0$ and $k_2 = k_3 = k_1$.

The temperature fields are described by

$$\frac{\partial^2 V_i}{\partial x^2} + \frac{\partial^2 V_i}{\partial y^2} = 0\,,\quad x,y \in C_i\,.$$

Besides the equations, the following boundary conditions are imposed. We have a heat flux at $x = -\delta$:

$$\left.\frac{\partial V_0}{\partial x}\right|_{x=-\delta} = -Q_0(y)\,.$$

Along the lines of symmetry, $y = 0$ and $y = l_0$ symmetry boundary conditions must be applied:

$$\frac{\partial V_i}{\partial n} = 0,$$

where n denotes the exterior normal to the boundary of the domains C_i. But at the other sides of the sample there is a heat exchange with the surrounding medium:

$$\frac{\partial V_i}{\partial n} + \beta_1^1 V_i = 0,$$

where $\beta_1^1 = \frac{h_1}{k_1}$.

Assuming that there is no contact resistance between the connected parts, we also add conjugation conditions:

$$V_i\Big|_{x=\dots} = V_j\Big|_{x=\dots}, \left.\frac{\partial V_i}{\partial x}\right|_{x=\dots} = \left.\frac{k_j}{k_i}\frac{\partial V_j}{\partial x}\right|_{x=\dots}, \; V_i\Big|_{y=\dots} = V_j\Big|_{y=\dots}, \left.\frac{\partial V_i}{\partial y}\right|_{y=\dots} = \left.\frac{k_j}{k_i}\frac{\partial V_j}{\partial y}\right|_{y=\dots}.$$

As the upper layer is quite thin, from now on we are going to assume that the temperature is uniform across the layer thickness. Hence from appropriate conjugation conditions we get these expressions:

$$V_2(x, y) = v_2(y) = V_0(0, y), \tag{4.1}$$

$$V_1(x, y) = v_1(x) = V(x, b), \tag{4.2}$$

$$V_3(x, y) = v_3(y) = V(l, y). \tag{4.3}$$

Because of that, we only need to solve the problem defined for the basic layer. So, we have the Laplace equations

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \tag{4.4}$$

$$\frac{\partial^2 V_0}{\partial x^2} + \frac{\partial^2 V_0}{\partial y^2} = 0 \tag{4.5}$$

with boundary conditions at the six sides. As we know from Section 2,

$$\left.\frac{\partial V_0}{\partial x}\right|_{x=-\delta} = -Q_0(y) \tag{4.6}$$

$$\left.\frac{\partial V_0}{\partial y}\right|_{y=0} = 0, \; \left.\frac{\partial V_0}{\partial y}\right|_{y=l_0} = 0, \tag{4.7}$$

$$\left.\frac{\partial V}{\partial y}\right|_{y=0} = 0, \tag{4.8}$$

But to get boundary conditions at $x = 0$, $x = l$, and $y = b$, we use appropriate conjugation conditions and expressions (1) – (3). For example, at $x = 0$ we have

$$\left.\left(\frac{\partial V_2}{\partial x}+\beta_1^1 V_2\right)\right|_{x=\varepsilon_0}=\frac{1}{k_1}\left.\left(k_0\frac{\partial V_0}{\partial x}+h_1 V_0\right)\right|_{x=0}=0$$

or

$$\left.\left(\frac{\partial V_0}{\partial x}+\beta_0^1 V_0\right)\right|_{x=0}=0\,,\ y\in(b,l_0)\,.$$

(4.9)

And

$$\left.\left(\frac{\partial V}{\partial x}+\beta_0^1 V\right)\right|_{x=l}=0\,,\ y\in(0,b)$$

(4.10)

$$\left.\left(\frac{\partial V}{\partial y}+\beta_0^1 V\right)\right|_{y=b}=0\,,\ x\in(0,l)$$

(4.11)

at $x=l\,,\ y=b\,.$

We also add conjugation conditions that state continuity of temperature and heat flux at the interface $x=0$:

$$V_0\big|_{x=-0}=V\big|_{x=+0}\,,$$

(4.12)

$$\left.\frac{\partial V_0}{\partial x}\right|_{x=-0}=\left.\frac{\partial V}{\partial x}\right|_{x=+0}\,.$$

(4.13)

Using conservative averaging method (see [10], etc.), we are going to construct an approximate solution for the given problem. Let's use an exponential approximation in the y -direction for the 2D temperature field $V(x,y)$ in the fin. The general form of the function is given by

$$V(x,y)=f_0(x)+\left(e^{\rho y}-1\right)f_1(x)+\left(1-e^{-\rho y}\right)f_2(x)\,,$$

(4.14)

where $\rho=b^{-1}$.

From symmetry condition (8) we find that $f_2(x)=-f_1(x)$. So, (14) assumes the following form:

$$V(x,y)=f_0(x)+2\left(\cosh(\rho y)-1\right)f_1(x)\,.$$

(4.15)

Defining the function $v(x)$ as integral average value

$$v(x)=\rho\int_0^b V(x,y)dy\,,$$

(4.16)

and integrating the expression (15) with respect to y, we can find the function $f_1(x)$:

$$f_1(x)=\frac{v(x)-f_0(x)}{2\left(\sinh(1)-1\right)}\,.$$

Let's substitute this in (14):

$$V(x,y) = \frac{\cosh(\rho y) - 1}{\sinh(1) - 1} v(x) + \frac{\sinh(1) - \cosh(\rho y)}{\sinh(1) - 1} f_0(x).$$
(4.17)

Applying the boundary condition (11), we have

$$\left(\rho \sinh(1) + \beta_0^1 (\cosh(1) - 1)\right) v(x) + \left(-\rho \sinh(1) + \beta_0^1 (\sinh(1) - \cosh(1))\right) f_0(x) = 0.$$

And hence

$$f_0(x) = \psi v(x)$$
(4.18)

with

$$\psi = \frac{\sinh(1) + \beta_0^1 b(\cosh(1) - 1)}{\sinh(1) + \beta_0^1 b(\cosh(1) - \sinh(1))}.$$

It follows immediately from (18) that

$$V(x,y) = v(x)\Phi(y),$$
(4.19)

where

$$\Phi(y) = \frac{\sinh(1) + \beta_0^1 b(\cosh(1) - \cosh(\rho y))}{\sinh(1) + \beta_0^1 b(\cosh(1) - \sinh(1))}.$$
(4.20)

Now the expression (14) (or (19)) contains only one unknown function – $v(x)$. In order to determine it, we use the definition (16) and from the partial differential equation (4) obtain an ordinary one for the unknown function:

$$\frac{d^2 v}{dx^2} + \rho \left.\frac{\partial V}{\partial y}\right|_{y=0}^{y=b} = 0, \quad x \in (0,l).$$

The difference of the derivatives may be found via the boundary conditions (8) and (11), and (19):

$$\frac{d^2 v}{dx^2} - \lambda^2 v(x) = 0,$$
(4.21)

where

$$\lambda^2 = \rho\beta_0^1 \Phi(b).$$

Applying the same operator (16) to (10) we get a boundary condition

$$v'(l) + \beta_0^1 v(l) = 0.$$
(4.22)

A solution to the problem (21), (22) is hence found to be

$$v(x) = c_1\left(e^{\lambda x} + \mu e^{-\lambda x}\right),$$
(4.23)

where

$$\mu = \frac{\lambda + \beta_0^1}{\lambda - \beta_0^1} e^{2\lambda l}$$

and c_1 is an unknown constant.

Therefore

$$V(x,y) = c_1\left(e^{\lambda x} + \mu e^{-\lambda x}\right)\Phi(y).$$
(4.24)

We act almost equally for the wall, and approximate the 2D temperature field $V_0(x,y)$ using exponential approximation in the x – direction:

$$V_0(x,y) = g_0(y) + \left(e^{-dx} - 1\right)g_1(y) + \left(1 - e^{dx}\right)g_2(y)\,, \qquad (4.25)$$

with $d = \delta^{-1}$.

Once again we use the properties of the function to solve for the unknown functions $g_i(y)$, $i = 0,1,2$.

We obtain average value function by the integral

$$v_0(y) = d\int_{-\delta}^{0} V_0(x,y)dx\ .$$

(4.26)

Integrating (25) over the segment $(-\delta, 0)$, gives

$$v_0(y) = g_0(y) + (e-2)g_1(y) + e^{-1}g_2(y)\ .$$

(4.27)

The function $V_0(x,y)$ also satisfies the boundary condition (6):

$$eg_1(y) + e^{-1}g_2(y) = \delta Q_0(y)\ .$$

(4.28)

So, combining (27) and (28), we have

$$g_1(y) = \frac{1}{2}\left(g_0(y) - v_0(y) + \delta Q_0(y)\right).$$

(4.29)

Finally, we conclude from (28) and (29) that formula (25) becomes

$$V_0(x,y) = g_0(y)\left(1 + \frac{1}{2}\left(e^{-dx} - 1\right) - e^2\frac{1}{2}\left(1 - e^{dx}\right)\right) + v_0(y)\left(-\frac{1}{2}\left(e^{-dx} - 1\right) + e^2\frac{1}{2}\left(1 - e^{dx}\right)\right)$$

$$+Q_0(y)\left(\frac{1}{2}\delta\left(e^{-dx} - 1\right) + +\left(\delta e - e^2\frac{1}{2}\delta\right)\left(1 - e^{dx}\right)\right).$$

(4.30)

Here we examine the part of the base that occupies the domain $x \in (-\delta, 0)$, $y \in (b, l_0)$.

Applying the boundary condition (9) to (30),

$$g_0(y)\left(-\frac{1}{2}d + \beta_0^1 + \frac{1}{2}de^2\right) + v_0(y)\left(\frac{1}{2}d - \frac{1}{2}de^2\right) + Q_0(y)\left(-\frac{1}{2} - e + \frac{1}{2}e^2\right) = 0\ .$$

We can rewrite this identity as

$$g_0(y) = v_0(y)a_0 + Q_0(y)b_0\,,$$

(4.31)

where

$$a_0 = \frac{d - de^2}{d - 2\beta_0^1 - de^2}\,,\ b_0 = -\frac{1 + 2e - e^2}{d - 2\beta_0^1 - de^2}\ .$$

Substituting (31) into the representation (30), yields

$$V_0(x,y) = v_0(y)\left(a_0 + \frac{1}{2}(a_0 - 1)\left(e^{-dx} - 1\right) + e^2\frac{1}{2}(1 - a_0)\left(1 - e^{dx}\right)\right)$$

$$+Q_0(y)\left(b_0 + \frac{1}{2}(b_0 + \delta)\left(e^{-dx} - 1\right) + \left(\delta e - e^2\frac{1}{2}(\delta + b_0)\right)\left(1 - e^{dx}\right)\right).$$

(4.32)

This shows that the function now depends only on one unknown - function $v_0(y)$. Let's integrate the main equation (5) in the x-direction

$$d\left.\frac{\partial V_0}{\partial x}\right|_{x=-\delta}^{x=0}+\frac{d^2 v_0}{dy^2}=0 .$$

(4.33)

The first addend can be found directly from the boundary condition (6), and (32). So (33) results in an ordinary differential equation

$$\frac{d^2 v_0}{dy^2}-\kappa^2 v_0(y)=\gamma Q_0(y), \quad y\in(b,l_0)$$

(4.34)

where

$$\kappa^2=-\frac{1}{2}d^2\left(1-e^2\right)\left(1-a_0\right)=d\beta_0^1 a_0, \gamma=-\frac{1}{2}d\left(b_0 d\left(e^2-1\right)+\left(1-e\right)^2\right).$$

For simplicity reasons we henceforth assume the function $Q_0(y)$ to be constant, that is $Q_0(y)=Q_0$.

Integrating (7_2), we obtain a boundary condition:

$$v_0'(l_0)=0 .$$

(4.35)

The solution of (34), (35) is

$$v_0(y)=c_2\left(e^{\kappa y}+\mu_0 e^{-\kappa y}\right)-\frac{\gamma Q_0}{\kappa^2}$$

(4.36)

with

$$\mu_0=e^{2\kappa l_0}$$

and c_2 as an unknown constant.

To find the equation for the left part of the base, where $y\in(0,b)$, let us remind you that for $x=0$ the functions $V_0(x,y)$, $V(x,y)$ satisfy the conjugation conditions (12), (13). So, from (13) and (24) it follows that

$$d\left.\frac{\partial V_0}{\partial x}\right|_{x=-0}=d\left.\frac{\partial V}{\partial x}\right|_{x=+0}=dc_1\lambda(1-\mu)\Phi(y).$$

(4.37)

If we now use (37) and the boundary condition (6), then equation (33) becomes

$$\frac{d^2 v_0}{dy^2}=-dc_1\lambda(1-\mu)\Phi(y)-dQ_0, \quad y\in(0,b).$$

Let us rewrite it as

$$\frac{d^2 v_0}{dy^2}=-c_1 B_1-dQ_0+c_1 B_2\cosh(\rho y),$$

(4.38)

where

$$B_1=d\lambda(1-\mu)\Phi_0, \quad B_2=\beta_0^1 bd\lambda(1-\mu)\Phi_1,$$

$$\Phi_1=\left(\sinh(1)+\beta_0^1 b\left(\cosh(1)-\sinh(1)\right)\right)^{-1}, \ \Phi_0=\left(\sinh(1)+\beta_0^1 b\cosh(1)\right)\Phi_1 .$$

From (7_1) we get a boundary condition:

$v_0'(0) = 0$.

(4.39)

So, the solution of the problem (38), (39) is

$$v_0(y) = c_1 B_2 \cosh(\rho y) b^2 + \frac{1}{2}(-c_1 B_1 - dQ_0) y^2 + c_3 .$$

(4.40)

When finding temperature for the left part of the base, one should take into account, that the function $g_0(y)$ is still unknown. We find that from the conjugation condition (12). Indeed, putting $x = 0$ in (30) and (24), we get that

$$g_0(y) = c_1(1+\mu)\Phi(y) .$$

(4.41)

We have just found solution to the given problem. But we are still left with finding the unknown constants in the formulas (23), (36) and (40). To determine those, we will need several requirements to be fulfilled. First, the temperatures $V_0(x, y)$, $V(x, y)$ must coincide at the contact point $x = 0, y = b$ between the fin and the right part of the wall, so

$$c_1(1+\mu)\Phi(b) = c_2 a_0 \left(e^{\kappa b} + \mu_0 e^{-\kappa b}\right) - a_0 \frac{\gamma Q_0}{\kappa^2} + Q_0 b_0$$

(4.42)

Second, the mean temperature values in the wall have continuity at $y = b$:

$$c_2\left(e^{\kappa b} + \mu_0 e^{-\kappa b}\right) - \frac{\gamma Q_0}{\kappa^2} = c_1\left(B_2 \cosh(1) b^2 - \frac{1}{2} B_1 b^2\right) - \frac{1}{2} d Q_0 b^2 + c_3 .$$

(4.43)

Third, we claim that the mean fluxes also coincide at $y = b$:

$$c_2 \kappa \left(e^{\kappa b} - \mu_0 e^{-\kappa b}\right) = c_1\left(B_2 \sinh(1) b - B_1 b\right) - d Q_0 b .$$

(4.44)

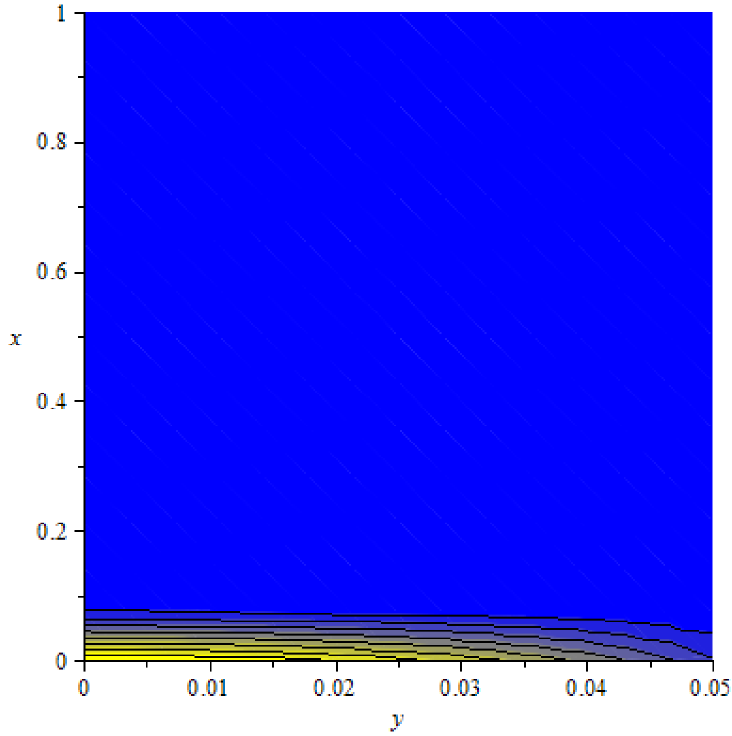
All the constants can be found from the system (42), (43), and (44). So, the approximate analytical solution to the problem is uniquely determined.

To get numerical results for temperature distribution in the sample, we used the following geometrical parameters:

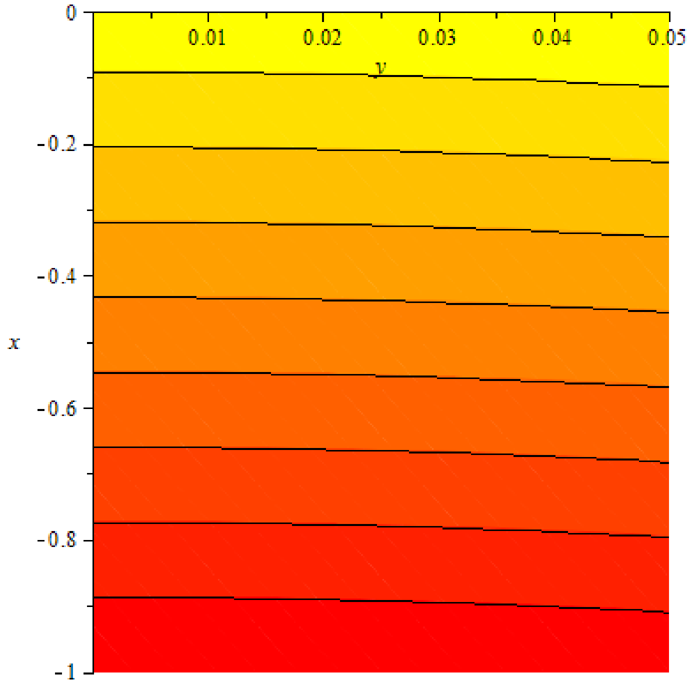
$\delta = 500\ \mu m$, $l = 1 \mu m$, $b = 5 \cdot 10^{-2}\ \mu m$, $l_0 = 1 \cdot 10^{-1}\ \mu m$.

But for the term physical properties we chose:

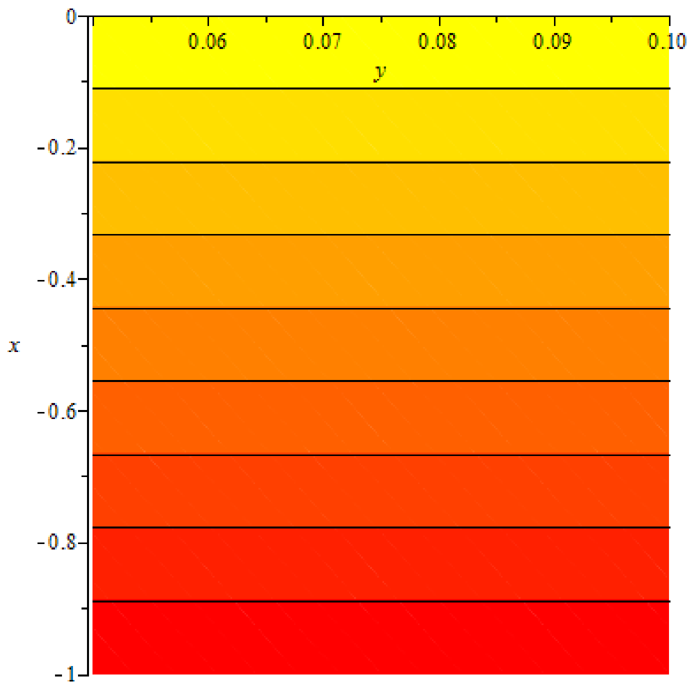
$h_1 = 11.085 \cdot 10^{-4} W \mu m^{-2} K^{-1}$, $k_0 = 9 \cdot 10^{-6} W \mu m^{-1} K^{-1}$, $Q_0 = 50\ K \mu m^{-1}$.



Temperature distribution in the fin



Temperature distribution in the left side of the base



Temperature distribution in the right side of the base

5. Steel quenching process

Contrary to the traditional method the intensive quenching process uses environmentally friendly highly agitated water or low concentration of water/mineral salt solutions and very fast cooling rates are applied. We propose to use hyperbolic heat equation for more realistic description of the intensive quenching process (especially for process initial stage). Here we consider few other models and construct solutions for direct and inverse problems of hyperbolic heat conduction equation [15].
The non-dimensional temperature field fulfils hyperbolic heat equation (telegraph equation):

$$\tau_r \frac{\partial^2 V}{\partial t^2} + \frac{\partial V}{\partial t} = a^2 \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \right), x \in (0, l), y \in (0, b), z \in (0, w), t \in (0, T], \quad a^2 = \frac{k}{c\rho}.$$

(5.1)

Here c is the specific heat capacity, k - the heat conduction coefficient, ρ - the density, τ_r - the relaxation time. It is natural assumption that planes $x = 0, y = 0, z = 0$ are symmetry surfaces of the sample:

$$\left.\frac{\partial V}{\partial x}\right|_{x=0} = \left.\frac{\partial V}{\partial y}\right|_{y=0} = \left.\frac{\partial V}{\partial z}\right|_{z=0} = 0.$$

(5.2)

On the all other sides of steel part we have heat exchange with environment. Although the method proposed here is applicable for non-homogeneous environment temperature, for simplicity we consider models of constant environment temperature $\Theta_0 = 0$. This restriction gives following homogeneous third type boundary conditions on the all three outer sides. The initial conditions are assumed in form:

$$V|_{t=0} = V_0(x, y, z),$$

(5.3)

$$\left.\frac{\partial V}{\partial t}\right|_{t=0} = W_0(x, y, z).$$
(5.4)

From the practical point of view the condition (4) is unrealistic. The initial heat flux must be determined theoretically. As additional condition we assume that the temperature distribution and the heat fluxes distribution at the end of process are given (known):

$$V\big|_{t=T} = V_T(x, y, z),$$
(5.5)

$$\left.\frac{\partial V}{\partial t}\right|_{t=T} = W_T(x, y, z).$$
(5.6)

As the first step we use well known substitution: $V(x, y, z, t) = \exp\left(-\frac{t}{2\tau_r}\right)U(x, y, z, t).$

Then the differential equation (1) transforms into differential equation without first time derivative:

$$\frac{\partial^2 U}{\partial t^2} = a_\tau^2\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\right) + \frac{1}{4\tau_r^2}U,\ x \in (0, l),\ y \in (0, b),\ z \in (0, w), t \in (0, T],$$

$$a_\tau^2 = a^2/\tau_r.$$
(5.7)

The initial and boundary conditions take the form:

$$U\big|_{t=0} = V_0(x, y, z),$$
(5.8)

$$\left.\frac{\partial U}{\partial t}\right|_{t=0} = W_0(x, y, z) + \frac{V_0(x, y, z)}{2\tau_r},$$
(5.9)

$$\left.\frac{\partial U}{\partial x}\right|_{x=0} = \left.\frac{\partial U}{\partial y}\right|_{y=0} = \left.\frac{\partial U}{\partial z}\right|_{z=0} = 0,$$
(5.10)

$$\left.\left(\frac{\partial U}{\partial x} + \beta U\right)\right|_{x=l} = \left.\left(\frac{\partial U}{\partial y} + \beta U\right)\right|_{y=b} = 0,$$
(5.11)

$$\left.\left(\frac{\partial U}{\partial z} + \beta U\right)\right|_{z=w} = 0.$$
(5.12)

Additional conditions transform as follow:

$$U\big|_{t=T} = \exp\left(\frac{T}{2\tau_r}\right)V_T(x, y, z),$$
(5.13)

$$\left.\frac{\partial U}{\partial t}\right|_{t=T} = \exp\left(\frac{T}{2\tau_r}\right)\left[W_T(x, y, z) + \frac{V_T(x, y, z)}{2\tau_r}\right].$$
(5.14)

We will start with formulation of the mathematical model for thin in y, z – directions of steel part (one-dimensional model): $w \ll l, b \ll l$. Then in accordance with conservative averaging method we introduce following integral averaged value:

$$u(x,t) = (bw)^{-1} \int_0^b dy \int_0^w U(x,y,z,t)dz. \tag{5.15}$$

Assuming the simplest approximation by constant in the y, z – directions, we obtain 1-D differential equation with the source term:

$$\frac{\partial^2 u}{\partial t^2} = a_\tau^2 \frac{\partial^2 u}{\partial x^2} - cu, x \in (0,l), t \in (0,T], \ c = \left[\beta \left(\frac{1}{b} + \frac{1}{w} \right) - \frac{1}{4\tau_r^2} \right]. \tag{5.16}$$

Initial conditions (13), (14) for the differential equation (16) are as follow:

$$u\big|_{t=0} = u_0(x), \quad u_0(x) = (bw)^{-1} \int_0^b dy \int_0^w V_0(x,y,z)dz,$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = v_0(x), v_0(x) = w_0(x) + \frac{u_0(x)}{2\tau_r}, \ w_0(x) = (bw)^{-1} \int_0^b dy \int_0^w W_0(x,y,z)dz.$$

The boundary conditions remain in the same form. Solution of this one-dimensional direct problem is well known:

$$u(x,t) = \frac{\partial}{\partial t} \int_0^l u_0(\xi) G(x,\xi,t) d\xi + \int_0^l v_0(\xi) G(x,\xi,t) d\xi. \tag{5.17}$$

The Green function has representation:

$$G(x,\xi,t) = \sum_{i=1}^{m-1} \frac{\varphi_i(x)\varphi_i(\xi)\sinh\left(t\sqrt{\left|a_\tau^2\lambda_i^2 + c\right|}\right)}{\|\varphi_i\|^2 \sqrt{\left|a_\tau^2\lambda_i^2 + c\right|}} + \sum_{i=m}^{\infty} \frac{\varphi_i(x)\varphi_i(\xi)\sin\left(t\sqrt{a_\tau^2\lambda_i^2 + c}\right)}{\|\varphi_i\|^2 \sqrt{a_\tau^2\lambda_i^2 + c}},$$

$$\varphi_i(x) = \cos(\lambda_i x), \ \|\varphi_i\|^2 = \frac{l}{2} + \frac{\beta}{2(\lambda_i^2 + \beta^2)}.$$

Here the natural number m in the both sums is given by inequalities:

$$a_\tau^2\lambda_i^2 + \left[\beta \left(\frac{1}{b} + \frac{1}{w} \right) - \frac{1}{4\tau_r^2} \right] < 0, i = \overline{1, m-1}, \ a_\tau^2\lambda_i^2 + \left[\beta \left(\frac{1}{b} + \frac{1}{w} \right) - \frac{1}{4\tau_r^2} \right] > 0, i = \overline{m, \infty}.$$

The eigenvalues λ_i are roots of the transcendental equation: $\lambda \tan(\lambda l) = \beta$.

As it was told earlier, from experimental point of view second initial condition is unrealizable and the $v_0(x)$ must be calculated theoretically. The differentiation of solution gives:

$$\frac{\partial}{\partial t} u(x,t) = \int_0^l u_0(\xi) \frac{\partial^2}{\partial t^2} G(x,\xi,t) d\xi + \int_0^l v_0(\xi) \frac{\partial}{\partial t} G(x,\xi,t) d\xi. \tag{5.18}$$

There is an interesting situation, if both additional conditions are known. In this case we introduce new time argument by formula

$$\tilde{t} = T - t. \tag{5.19}$$

The main differential equation remains its form and the boundary conditions remain the same. Both additional conditions transform to initial conditions for the equation:

$$u\big|_{\tilde{t}=0} = u_T(x),\ \frac{\partial u}{\partial \tilde{t}}\bigg|_{\tilde{t}=0} = -v_T(x).$$

(5.20)

The solution of direct problem (29), (22) and (30) is similar with the solution. For the heat flux we have a nice explicit representation for the initial heat flux:

$$v_0(x) = \int_0^l u_T(\xi)\frac{\partial^2}{\partial \tilde{t}^2}G(x,\xi,\tilde{t})\Big|_{\tilde{t}=T} d\xi - \int_0^l v_T(\xi)\frac{\partial}{\partial \tilde{t}}G(x,\xi,\tilde{t})\Big|_{\tilde{t}=T} d\xi.$$

(5.21)

By applying conservative averaging method to the problem we obtain relatively the integral average temperature $u_0(t)$ following boundary problem for ordinary differential equation:

$$\tau_r\frac{du_0^2}{dt^2} + \frac{du_0}{dt} + \frac{h}{c_H}u_0 = \frac{h}{c_H}\Theta(t) + f(t), c_H = c\rho H, u_0(0) = u_0^0, u_0(T) = u_T.$$

(5.22)

We are interested to determine

$$v_0 = \frac{du_0(0)}{dt}.$$

(5.23)

Here $\beta = \dfrac{1}{2\tau_r}\sqrt{1 - 4\cdot\tau_r\cdot\dfrac{h}{c_H}}$. (44)

Consequently, we have finally obtained the solution of this problem as:

$$u_0(t) = e^{-\frac{t}{2\tau_r}}[u_0^0 e^{\frac{t}{2\tau_r}\beta} + (\frac{1}{2\tau_r}u_0^0 - u_0^0\frac{1}{\beta 2\tau_r} - \frac{v_0}{\beta})\sinh(\frac{t}{2\tau_r}\beta)]$$

$$+\frac{2\tau_r}{\beta}e^{-\frac{t}{2\tau_r}}\int_0^t \sinh(\frac{(t-\omega)}{2\tau_r}\beta)(e^{\frac{\omega}{2\tau_r}}\cdot\frac{h}{c\cdot\rho}\Theta(\omega))d\omega.$$

As the last step we use the additional information – condition (20), i.e. the known value at the end of the process. This information allows us to express unknown second initial condition in closed and simple form:

$$v_0 = \frac{\beta}{\sinh(\frac{T\beta}{2\tau_r})}(u_T\cdot e^{\frac{T}{2\tau_r}} + u_0^0\cdot e^{\frac{T}{2\tau_r}\beta} + \frac{u_0^0}{2\tau_r}(1-\frac{1}{\beta})\sinh(\frac{T\beta}{2\tau_r}))$$

$$+\frac{2\tau_r}{\sinh(\frac{T\beta}{2\tau_r})}\int_0^T \sinh(\frac{(T-\omega)}{2\tau_r}\beta)\left(\frac{h}{c\cdot\rho}\Theta(\omega) + f(\omega)\right)e^{\frac{\omega}{2\tau_r}}d\omega.$$

(5.24)

We can increase the order of the approximation for the solution of the original problem by the representation with polynomial of second degree and exponential approximation. The integration over interval $x \in [0, H]$ of the main equation practically gives the same ordinary differential equation. The only difference is in the same coefficient at two terms. The additional conditions remain the same. It means that we can use obtained above formulae replacing the parameters β, γ by following expressions:

$$\beta = \frac{1}{2\tau_r}\sqrt{1 - \frac{4\cdot\tau_r\cdot h}{c_H(1+\frac{hR}{2k})}},\ \gamma = \frac{h}{c_H\left(1+\frac{hH}{2k}\right)}$$

(5.25)

We have obtained solution of well posed problem in closed form. This solution can be used as initial approximation for integrated over $x \in [0, H]$ equation.

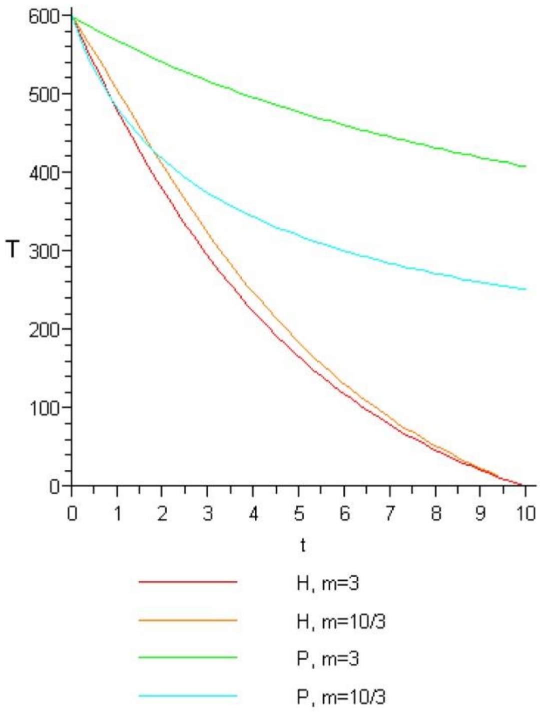
Conservative averaging method can be applied to problems with non linear BC. Condition for nucleate boiling $(m \in [3; 3\frac{1}{3}])$: $k\frac{\partial U}{\partial x} + \beta^m [U - \Theta_B(t)]^m = 0, x = R, t \in [0, T]$. We solved several problems and obtained numerical results using Maple and COMSOL Multiphysics. Modeling is done for a silver ball, r=0.02m, temperature at t=0 is 600°C, at t=T 0°C.

Dependence on τ value

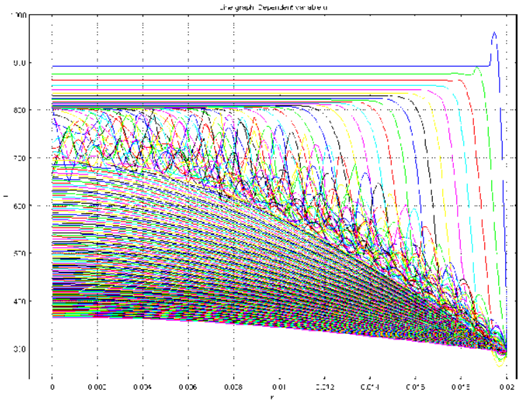
τ_r	v_0
0.2	-4499.270429
0.5	-1799.270299
1.5	-600.2998684

If we compare solutions of classic – parabolic – and hyperbolic heat conduction problems, using nonlinear boundary condition case, we obtain graphic:

.

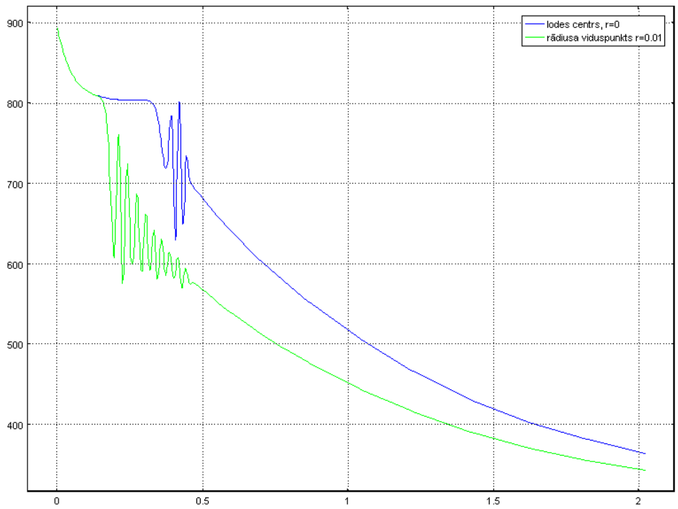


We examined temperature on the radius. As you can see, the temperature on radius is not monotony. It means that the form of boundary condition on the surface can vary:



Temperature distribution on radius

It is very clear that at the beginning of the process hyperbolic term is extremely important but later process is described by classic heat equation. It is possible to define precise points were temperature is computed.



Temperature changes at r=0m and r=0.01m

References

[1] Editors: **Myriam Lazard, Andris Buikis, e.a**. *Recent Advances in Fluid Mechanics and Heat & Mass Transfer*, Proceedings of the 9th IASME/WSEAS Internacional Conference on Fluid Mechanics & Aerodynamics and Heat Transfer, Thermal Engineering and Envinment. Florence, Italy, August 23-25, 2011. WSEAS Press. (Editors' forward –p. 7.)

[2] Editors: **Myriam Lazard, Andris Buikis, e.a.** *Recent Advances in Applied & Biomedical Informatics and Computational Engineering in System Applications.* Proceedings of the 4th WSEAS International Conference on Biomedical Electronics and Biomedical Informatics (BEBI'11). Florence, Italy, August 23-25, 2011. WSEAS Press.

[3] Editors: **Myriam Lazard, Andris Buikis, e.a**. *Recent Advances in Signal Processing, Computational Geometry and Systems Theory.* Proceedings of the 11th WSEAS International Conference on Signal Processing, Computational Geometry and Artificial Vision. Proceedings of the 11th WSEAS International Conference on System Theory and Scientific Computation. Florence, Italy, August 23-25, 2011. WSEAS Press.

[4] **Buikis, A.** *Interpolation of integral mean of piecewise smooth function by means of parabolic spline.* Latvian Mathematical Yearbook, 1985, No. 29, pp. 194-197. (In Russian)

[5] **Buikis, A**. *Calculation of coefficients of integral parabolic spline.* Latvian Mathematical Yearbook, 1986, No. 30, pp. 228-232. (In Russian)

[6] **M.Buike, A.Buikis.** *Analytical Approximate Method for Three-Dimensional Transport Processes in Layered Media.* Proceedings of 4th IASME/WSEAS International Conference on HEAT TRANSFER, THERMAL ENGINEERING and ENVIRONMENT, Elounda, Crete Island, Greece, August 21-23, 2006, pp. 232-237.

[7] **M. Buike, A. Buikis**. *System of Models for Transport Processes in Layered Strata*. Proceedings of 5th WSEAS International Conference on SYSTEM SCIENCE and SIMULATION in ENGINEERING, Tenerife, Canary Islands, Spain, December 16-18, 2006, pp. 19-24.

[8] **Buikis, A.** *Definition and calculation of a generalized integral parabolic spline*. Proceedings of the Latvian Academy of Sciences. Section B, 1995, No. 7/8 (576/577), pp.97-100.

[9] **S.Kostjukova, A.Buikis**. *Integral rational spline with jump for discontinuous mathematical problems in layered media*. LATEST TRENDS ON **THEORETICAL** AND APPLIED MECHANICS, FLUID MECHANICS AND HEAT & MASS TRANSFER Proceedings of 5th IASME/WSEAS International Conference on Continuum Mechanics (CM'10), University of Cambridge, UK, February 23-25, 2010. pp. 279-282.

[10**]** **A. Buikis,** *Conservative averaging as an approximate method for solution of some direct and inverse heat transfer problems*. Advanced Computational Methods in Heat Transfer, IX, WIT Press, 2006, pp. 311-320.

[12] **S. Kostjukova, M. Buike, A. Buikis**, *Conservative averaging method for the heat conduction and convection processes in layered media.* The 6th Baltic Heat Transfer Conference (BHTC2011), August 24-26, 2011, Tampere, Finland. 6 p. (CD-ROM form).

[13] **M. Lencmane, A. Buikis**. *Analytical Two - Dimensional Solution for Transient Process in the System with Rectangular Fins*, Proceedings of the 6th International Scientific Colloquium, Riga, 2010, pp 157-180.

[14] **T. Bobinska, M. Buike, A. Buikis, H.H. Cho**. *Stationary heat transfer in system with double wall and double fins.* Advances in Data Networks, Communications, Computers and Materials. 5th WSEAS International Conference on Materials Science. Sliema, Malta, September 7-9, 2012. pp. 260-265.

[15] **S.Blomkalna, M.Buike, A.Buikis**. *Several Intensive Steel Quenching Models for Rectangular and Spherical Samples.* Recent Advances in Fluid Mechanics and Heat@Mass Transfer. Proceedings of the 9th IASME/WSEAS International Conference on THE'11. Florence, Italy, August 23-25, 2011. p. 390-395.