RESEARCH IN ELEMENTARY MATHEMATICS

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Target and planned tasks of the sub-activity

In order to secure long-term development of science, one needs to care for the training of the relevant new staff in Latvia. It cannot begin only at high-schools. Therefore, activities widening and deepening the Latvian pupils' knowledge and skills in mathematics, preparing the most gifted from them for as early as possible involvement in scientific research are significant.

Almost for 40 years already, A. Liepa's Extramural School of Mathematics (ESM) of the University of Latvia continues scientific work in the field and implements the obtained results. As a result of the research, a new branch of mathematics has appeared: the modern elementary mathematics (MEM) was included on the list of the official science classification in Latvia in 1995.

A study programme is developed within the framework of ESM scientific and pedagogical activities; it has included about 30 000 pupils so far. As a result of its activities:

1) a large number of applicants excellently prepared in mathematics enter the high-schools of Latvia every year;

2) many of them are prepared to be involved in scientific research already at the beginning of their studies;

3) scientifically-pedagogical teams receive qualitative supplement to their staff regularly thus promoting the development of science and improving the export of scientific production.

It was planned during the implementation of the Project:

1) to continue the research started in the modern elementary mathematics;

2) to improve and widen the set of pedagogic methods based on the mentioned research and oriented towards pupils, teachers and the students – future teachers;

3) to reflect the obtained results in internationally reviewed articles, papers read on international conferences, and textbooks.

1. Research and results in the area of advanced mathematics education

The main task of the work group was directed towards the promotion of the youth's interest in the sciences, especially mathematics, the development of skills in mathematics, and the development of pupils' skills and abilities in research.

Unfortunately, greater and greater attention is paid only to practical use of the specific knowledge during the process of learning at school, while the meaning of proof is left neglected. In order to develop the pupils' skills of reasoning and substantiate their ideas precisely, as well as to explain the proofs in written, all tasks of competitions must offer full solutions with expanded explanations (the exception is the first stage of the competition for the grade 4 "*So many.. Or, how much?*", where the warm-up offers also questions about problems with multiple-choice answers provided). Due to this, one set includes a small number of problems usually while the time for their solution is relatively long (the mathematics olympiads offer 5 problems and 5 hours to solve them). A regular demand for the proofs to one's conclusions develops also an inner need for to substantiate any thought or statement – it is essential not only for researchers, but also for any well furnished mind. The general methods of combinatorics play an essential role in the development of judging and substantiation skills (including also the skills in research). They are based on generally accepted findings of the humanity acquired during hundreds of years therefore their essence is easily grasped at all pupils' ages. These methods are not only the techniques for the solution of one specific problem or a group of problems, but they are widely applicable for many situations of life, including the higher mathematics.

1) The essence of the ***method of invariants*** is to search for invariable amounts or features in some process or combination. The method of invariants is applied mainly to the evidences of impossibility.

2) The ***the mean value method*** may be expressed in the following words: „*in order to do great things it is necessary to concentrate large resources at least for one direction.*" The special case of the method of average value is applied often – *Dirichlet principle*: *if more than n rabbits must be placed in n cages, at least one of the cages will hold at least two rabbits*. The method of mean value is applicable to substantiate the existence of some kind of objects.

3) The ***method of the extreme element*** searches for the extreme (in some sense) element of a set, and the conclusions about the properties of such set are made on the basis of its properties.

4) The ***method of mathematic induction*** is a widely used technique of mathematical proofs, where general judgements are based on specific judgements.

5) The ***method of interpretations*** is a general technique to solve a problem, when the given problem of a sub-branch is "translated" (interpreted) into the "language" of another sub-branch, solved and the solution is "translated" back, thus acquiring the solution for the given problem, or the problem is replaced by an isomorphic problem during the application of the interpretation method, and such problem is solved then.

During the project, the main attention was paid to the mastering of these methods and their application.

2. Activities for deeper studies of mathematics and for the promotion of pupil's interest in mathematics

2.1. Mathematical competitions

The spirit of competition is one of the motivators expressed especially well during the younger age of school and for teenagers. As the experience shows, contests serve as a successful method for deeper studies. In mathematics, these are all kinds of competitions and olympiads. In order to achieve good results in competitions, one needs regular training on his own or guided by a teacher of a mathematics hobby group or optional lessons. Traditionally, several contests are held for all groups of pupils and students in mathematics in Latvia. The existing traditions were continued during the accounting period by improving the order and content of the contests and by providing problems with more modern contents. Taking into consideration the reduced standards for the mathematics in schools, the problems are developed so as to exclude the specific mathematical facts not meant for and not taught to the relevant class, but the focus might be placed on creative and untraditional application of the techniques of reasoning.

1. Extramural contests for primary school pupils

1.1. „*Young Mathematicians' Contest*" is a contest of doing mathematical problems for the pupils of the grades 4 to 7. During the school year, five stages are held; they offer 5 tasks to the pupils at each stage. During the accounting period, 15 sets of problems were developed, 75 problems in total, their solutions found, also the analysis of the pupils' results is carried out. During the school year, ~250 participants are involved in the contest.

1.2 „*Professor Littledigit's Club*" is a contest of doing mathematical problems for the pupils of the classes 5 to 9. It has traditions for 39 years already, and the pupils' and their teachers' trust is won by it. Untraditional, research-oriented problems are offered already to the pupils of the younger school age. During the accounting period, sets of problems were developed for 3 school years, i.e., 18 sets of problems containing 180 problems in total, also the solutions for the problems were established and the pupils' results analysed. About ~ 200 participants are involved during a school year.

2. The contest – Mathematical Olympiad for the grade 4 "*So many.. Or, how much?*" is held already for the tenth year in Latvia, and its popularity grows with each year. ~3500 pupils from ~170 schools of Latvia were involved in the contest during the previous year. The aim of the contest is to offer untraditional problems developing skills in logic and reasoning to the primary school pupils. 12 sets of problems, 108 problems in total, their solutions and methodological materials for the assessment of the works were developed during the accounting period, because the works of the contest were verified at schools. The pupils' results were processed and analysed.

3. *The Mathematical olympiads* are competitions rich in traditions in Latvia and other places of the world. The Mathematical Olympiad for the pupils of the grade 5 to 12 is the most popular contest in the subjects of study in Latvia. Latvia organizes the country-scale in three stages, and the Open Mathematical Olympiad.

3.1. The proceedings of the 2nd (regional) and the 3rd (national) stage of the State Mathematical olympiad are regulated by the order drawn by the Ministry of Education and Science, the National Centre for Education (NCE) „On the proceedings of Olympics in the subjects of study", but the 1st stage (school) or preparatory olympiad is the initiative of A.Liepa's ESM for many years already. During the accounting period, sets of problems and methodological materials (120 problems in total) were developed for the preparatory stage of the olympiad within the framework of the project.

3.2. Any pupil of grade 5 to 12 may participate in the *Open Mathematical Olympiad*. During the previous years, ~3000 participants have taken part each year. The organization of the Open Mathematical Olympiad has been the initiative of A.Liepa's ESM already for 40 years, but the process of the drawing the sets of problems, the methodological materials, as well as the procession and analysis of the results was carried out within the framework of the project during the accounting period. 120 problems were developed, the works submitted by the pupils, as well as the results were processed and analysed during the accounting period. The analysis of the results displays, what topics impose the greatest difficulties to the pupils, therefore larger attention is paid to their mastering, including their incorporation in the materials of NNV and the development of a wider range of theoretical materials for the introductory parts of the collections of mathematical problems.

3.3. In order to assess the level of Latvian students' readiness in comparison to the international background, the teams of the Latvian students participate in the *International Mathematical Olympiad* (IMO), in the *International Mathematical Team Olympiad "The Baltic Way"* (BW). In 2012, the national girls team of Latvia took part in the *European Girls' Mathematical Olympiad* (EGMO) for the first time. In order to select participants for the international contests, several additional competitions are organized. During the accounting period, three competitions for selection of the participants of the Olympiad „Baltic Way" were held; 60 problems were developed for this purpose. Six contests were held for the opportunity to take part in the International Mathematical Olympiad; 30 problems were developed for this need. Also participants for the European Girls' Mathematical Olympiad were selected in a contest, 5 problems were developed for it.

Our pupils' results on international Olympiads evidence that the system of advanced mathematical education is in good shape in Latvia: 1 silver medal, 3 bronze medals and 8 honourable mentions were won on the International Mathematical Olympiad; the team of Latvia has gained the 4th place on the teams Olympiad „Baltijas Ceļš" in 2010, and the 2nd place in 2011; one bronze medal is won on the European Girls' Mathematical Olympiad during the accounting period.

2.2. Educational activities

As not all schools have sufficiently qualified teachers able to prepare pupils for the participation in the contests of mathematics, also educative events are held in parallel to the contests.

1. *Extramural lessons for the secondary school students* (NNV) are a extramural course for secondary school students in separate topics of mathematics overlooked at schools. During the accounting period, also three cycles of school years were implemented; in total, 12 lessons for two age groups – the students of the grades 9 to 12 and 11 to 12. Materials and problems on such topics as „*Method of invariants*", „*Methods of arrangement*", „*Vectors"*, „*Dirichlet principle*" and „*Method of mathematic induction*" were developed and improved; 240 problems were elaborated in total.

2. During the accounting period, the *Little University of Mathematics* (MMU) for secondary school students were re-established. MMU's beginnings may be found at the beginning of 1960s, when doc. Oto Treilibs organized lessons of this kind in 1965. With short interruptions, the courses of MMU are read each school year once or twice a month. Usually, 2 lectures are organized at a time, each 90 minutes long. Many leading Latvian mathematicians, lecturers, scientists and students of the University of Latvia have participated as listeners, as well as the lecturers of MMU. MMU lectures include most varied topics – these are the areas and topics of mathematics not covered by the school curriculum widely, incl. popular-scientific lectures on the topics of the higher mathematics, also deepened and widened review of the topics from the school curriculum.

After a break of two years, MMU recommenced its work „newly inspired" from the school year 2011/12. Its main aim was to promote the pupils' interest in mathematics, to stress the diversity of mathematics and the possibilities for its application to different areas of science and life, thus urging young people to choose studies in the University of Latvia, the Faculty of Mathematics and Physics. Also the old form of teaching – lectures involving only a blackboard and chalk – does not correspond to the demands of the youth today. A computer presentation is an obligatory demand for each lecture, and also the pupils' practical work is desirable. In order to stimulate young people to follow the theme of the lecture, feedback must not lack every time: a small test at the end of the lecture about the topics discussed during the day, and a home task that needs to be done by the next lecture.

At the end of the school year, certificates are awarded to the most active and diligent students coming to the lectures.

Although the course is intended mainly for the pupils of the grades 10 to 12, several pupils from the classes 8 to 9 and the teachers of mathematics attend them also, thus evidencing about the necessity and usefulness of such kind of lessons.

2.3. Informative system

All information related to MMU and other discussed activities may be found on the home page of A. Liepa's NMS at http://nms.lu.lv. During the accounting period, work for its maintenance and improvements was carried out.

In order to carry out quantitative research about the impact of some work groups' implemented activities upon the pupils' achievements in long term, work at the development and maintenance of a data basis of the pupils' mathematic achievements fostered by Extramural School of Mathematics was carried out during the accounting period.

2.4. Development of educational materials

Various educational and supportive materials play an important role in any educational process. A. Liepa's ESM issue several educational aids for pupils and teachers every year. The educational aids include the problems and their extended solutions from the contests in mathematics of one school year, including the international contests, where the pupils of Latvia have participated. The educational aids are meant also for pupils' individual studies therefore a separate chapter provides suggestions and short hints about the ways of doing sums before giving full resolutions.

Also a short summary of theory is important – it is a new and significant feature implemented to the developed educational aids as a result of the project.
During the accounting period, work at six new books was finished; the participants of the project group are their co-authors.

[1.] D. Bonka, S. Krauze, A. Šuste: *Jauno matemātiķu konkurss 2000.-2005. gadā.* Rīga: LU, 2011, 104 pp.

[2.] A. Andžāns, D. Bonka, Z. Kaibe, L. Zinberga: Matemātikas sacensības 4. –9. klasēm 2009./2010. mācību gadā. Rīga: LU, 2011, 117 pp.

[3.] A. Andžāns, M. Avotiņa, L. Freija: Matemātikas sacensības 9. –12. klasēm 2009./2010. mācību gadā. Rīga: LU, 2011, 125 pp.

[4.] A. Andžāns, M. Avotiņa, I. Opmane, Z. Ozola, M. Stupāne: „Profesora Cipariņa kluba" uzdevumi un atrisinājumi 1986.-1989. gadā. Rīga: LU, 2011, 114 pp.

[5.] M. Avotiņa, L. Freija: Matemātikas sacensības 9. –12. klasēm 2010./2011. mācību gadā. Rīga: LU, 2012, 156 pp.

[6.] D. Bonka, Z. Kaibe, L. Zinberga: Matemātikas sacensības 4. –9. klasēm 2010./2011. mācību gadā. Rīga: LU, 2012, 127 pp.

2.5. Students' scientific research

Scientific research projects (SRP) play an important role in the development of students' skills in research during the secondary school. Within the framework of the project, also separate problem groups were considered, coming mainly from the area of combinatorial geometry, successfully applicable as topics for the pupils' scientific research projects. It was reported also on a seminar organized by NCE „Development of Pupils' Scientific Research projects in mathematics".

3. Research in combinatorial geometry

Combinatorial geometry researches the mutual placement of figures, as well as figures on discrete (squared) plane. A wide group of combinatorial geometry problems is related to *polymino*. *Polymino* is a figure on plane obtained from square units by adding them each to other along their sides at full length. If a polymino consists exactly of *n* square units, it is called *n*-mino. The wider researched and discussed polyminoes are pentaminoes, see Fig. 1 (problems related to them are often found also in the sets of problems for competitions in mathematics).

The resolutions of many polymino-related problems are linked to the full resorting of cases performed mainly by computers. Thus, the most successful key to the resolution of a problem is the development of the most efficient algorithm and computer program for the resorting.

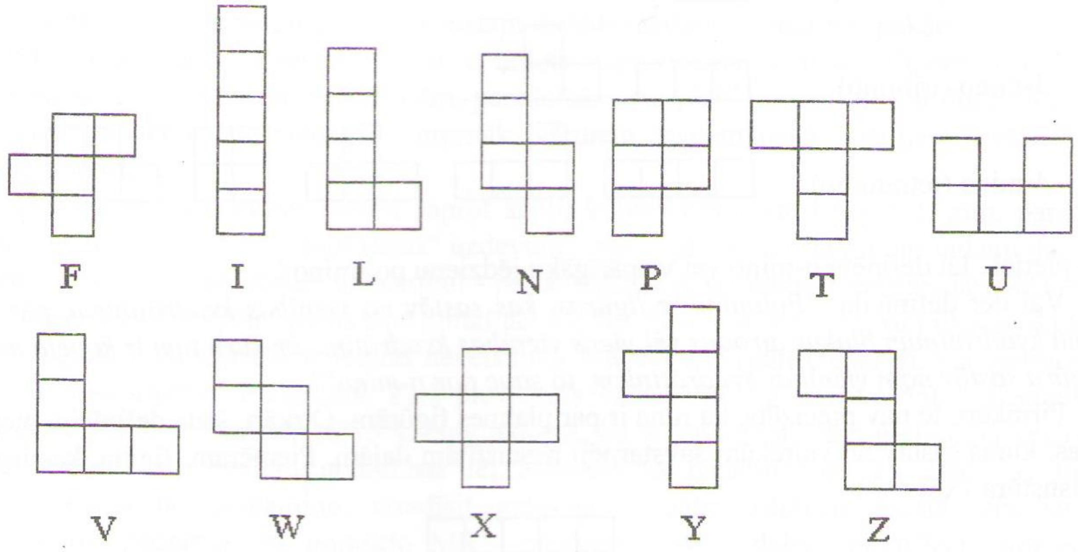


Fig. 1. Pentamionoes

3.1. Tetrads

Tetrads were the main object of research during the implementation of the project.

Definition. A *tetrad* is a figure in plane consisting of four equally sized polygons, each with other having a common border with a positive length.

Historical background. Tetrads are mentioned rarely in sources. Most probably, the notion of 'tetrad' has appeared for the first time in „*Journal of Recreational Mathematics*" in 1985. A small sub-chapter was devoted to tetrads in M. Gardner's book "*Penrose Tiles to Trapdoor Ciphers*" (1989). Several interesting tetrads consisting of different polyforms, including tetrads without holes were provided in this book. In 2008, K. Scherer elaborated a game offering to make tetrads from various polyforms within the framework of Wolfram Demonstrations Project. G. Sicherman placed an overview *Polyform Curiosities* about tetrads on a home page in 2011. Tetrads are studied by Juris Chernenoks in his master paper "*Some Unresolved Problems of Combinatorial Geometry*" (2012). Tetrads are studied also by the pupil of class 10 Anastasija Jakovleva, who has written a scientific research paper named "*Analysis of Tetrads*" (2012).

Problem 1. Find the smallest trend for each pentamino, which includes this pentamino as the only hole.

Algorithm for the construction of tetrads from polyminos was developed.

Algorithmic scheme. At the beginning, the list of polyminos is given, then each of them is analysed separately at each relevant cycle. All distributions in plane are found (these are more than 8). One copy of polymino is fixed. All ways are found, how to add the next polymino; a massive is developed from all these variants. All 3 sub-sets (a set consisting of three elements) of the massive are considered. If each two polyminos do not cover each other and they are situated side by side, this sub-set forms a tetrad together with the initially fixed polymino. The research of all sub-sets of three may be accomplished by the means of three sub-cycles, but optimization is included at this step. For example, when sub-sets $\{1, m, n\}$, $\{2, m, n\}$, ..., $\{m-1, m, n\}$, where $m < n$ are considered, then calculations are repeated for very many times to find, whether m-th and n-th polyminos do not cover each other and are situated side by side. When the principle of dynamic programming is used, such excessive calculations are prevented. In fact, this step is reduced to the finding of all such pairs of polyminos out of the whole massive, that are situated side to side to a polymino and do not cover each other.

It should be added, exactly the last step takes the most of the time during the analysis of each separate polymino, so the improvement of this step means significant economy in time.

Results. The developed algorithm was turned into a computer program. All possible tetrads consisting of n-minos, if $n \le 16$ were found by the program. The results are summarized by the table 1.

Tab. 1. Results of the Problem 1.

n	How many *n*-minoes form a tetrad	Number of tetrads
8	8	14
9	42	83
10	187	341
11	739	1388
12	2871	5648
13	11300	22688
14	44440	90243
15	172984	352163
16	670107	1373595

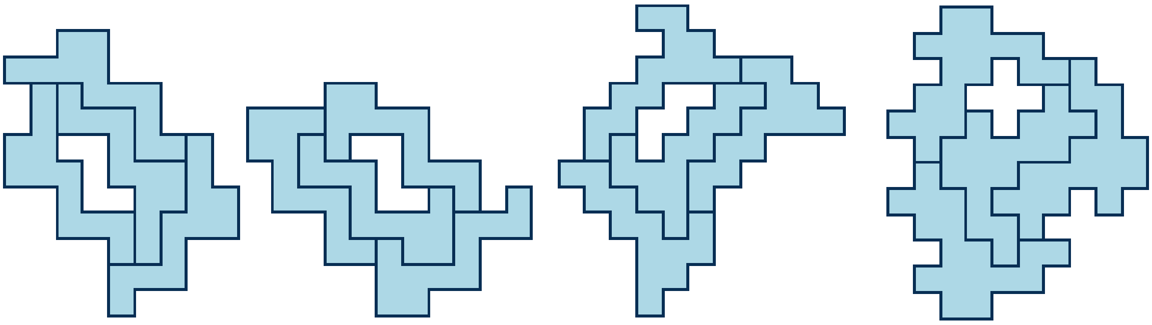
If a tetrad can be put together in several ways of one and the same polymino, then it is counted only for one time. About 115 hours were spent to find all tetrads of 16-mino.

By the analysis of the obtained data arrays of tetrads, the problem with smaller tetrads including each pentamino as the only hole was solved. The results were provided by the Table 2.

Tab. 2. The smallest tetrads including pentamino as the only hole.

Pentamino	**F**	**I**	**L**	**N**	**P**	**T**	**U**	**V**	**W**	**X**	**Y**	**Z**
n-mino, repesenting tetrad	15	10	12	15	10	15	15	15	13	17	15	12

Some examples are provided in the Fig. 2.



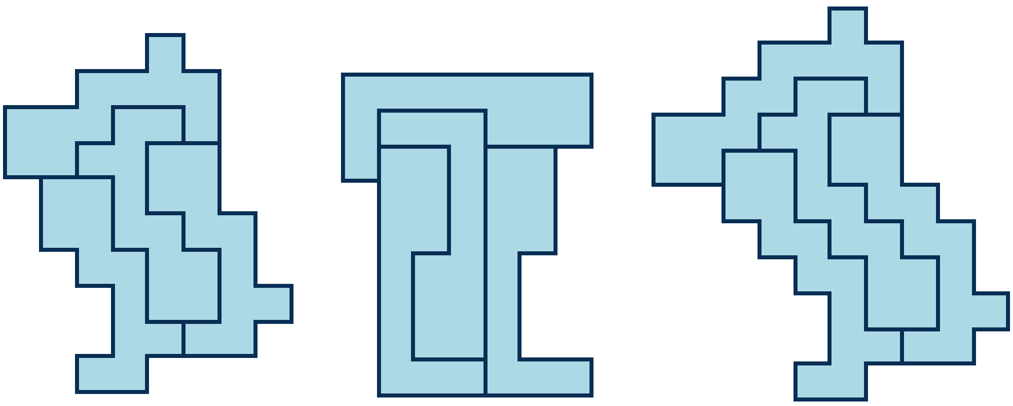
12-mino 13-mino 17-mino

Fig. 2.

Problem 2. Also the problem of existing tetrads without holes was studied. As a result of the research, the following theorem was formulated and proved.

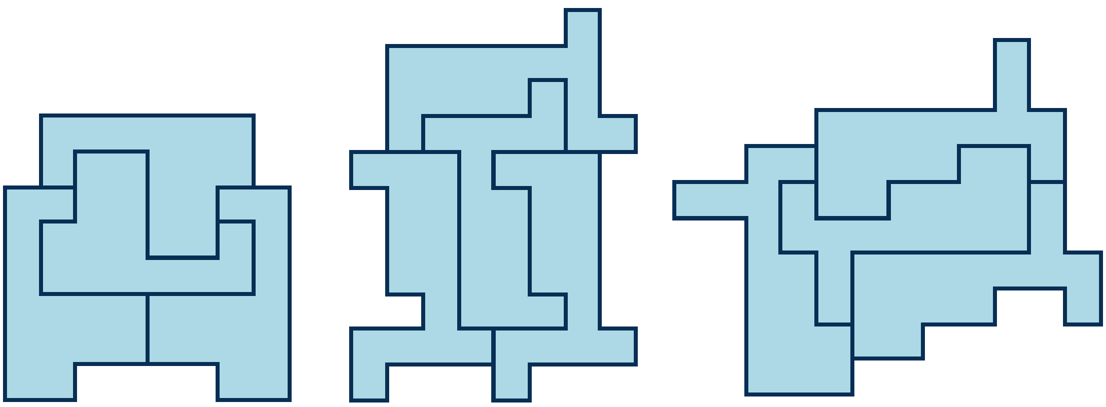
Theorem. Each n, $n \geq 11$ has n-mino existing that makes a tetrad without holes.

Proof. Tetrads without holes exist for 11, 12, 13, 14, 15 and 16-minos, and they are provided by the Fig. 3 and 4.



11-mino 12-mino 13-mino

Fig. 3.



14-mino 15-mino 16-mino

Fig. 4.

Now, let us see such 17-mino without holes, as shown in the Fig. 5.

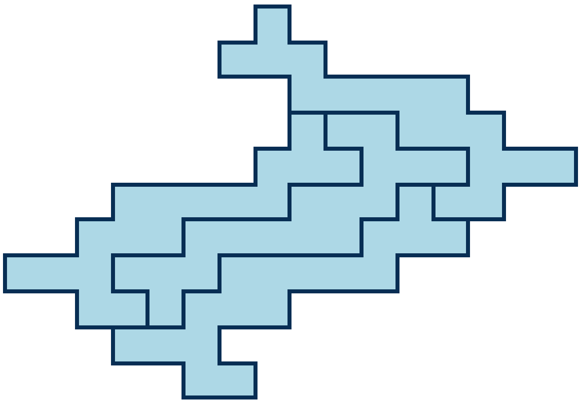


Fig. 5.

Upon closer inspection, the tetrad may be "extended" into horizontal direction as long as you like, as shown by the Fig. 6.:

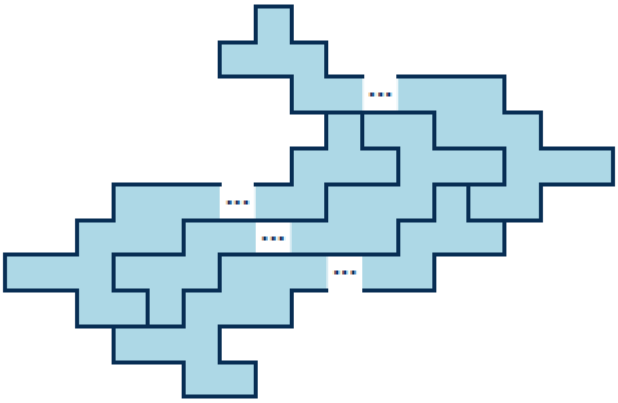


Fig. 6.

It becomes clear now that for n = 18, 19,… n-minos making tetrads without holes, also exist, what had to be proved.

Except polyminos, also tetrads consisting of such polyforms as *polymonds*, *polykites* and *polytans* were studied. A *polymond* is a plane figure obtained from several regular triangles adding them each to other at their whole-length sides. A *polykite* is a plane figure obtained from equal quadrilaterals (in particular, a kite) having angles 60°, 90°, 120° and 90° and two equal pairs of adjacent sides added each to other at their whole-length sides. A *polytan* a figure on plane obtained from several equal isosceles right triangles, when they are added each to other along their whole-length sides.

The computer program for the search of tetrads was transformed for the work with the mentioned polyforms for this purpose. For example, all tetrads were found, which can be make of n-kaits, if $n \leq 14$. The results are summarized by the Table 3.

Tab. 3. Tetrads from n-kites

n	How many n-kites form a tetrad	Number of tetrads
7	1	1

8	4	4
9	7	17
10	39	59
11	162	323
12	547	1020
13	1782	3216
14	6422	10970

As the Table 3 shows, the smaller tetrads are made of 7-kites, and such tetrad is one and only. Besides, it was found out that the smallest tetrad without holes is made of a 9-kite (see Fig. 7). Tetrads consisting of polykites are not mentioned in any of the known sources about tetrads. The obtained results were sent to G. Sicherman. In this way, some of the most interesting tetrads were added to the section of tetrads at the home page „*Polyform Curiosities*".

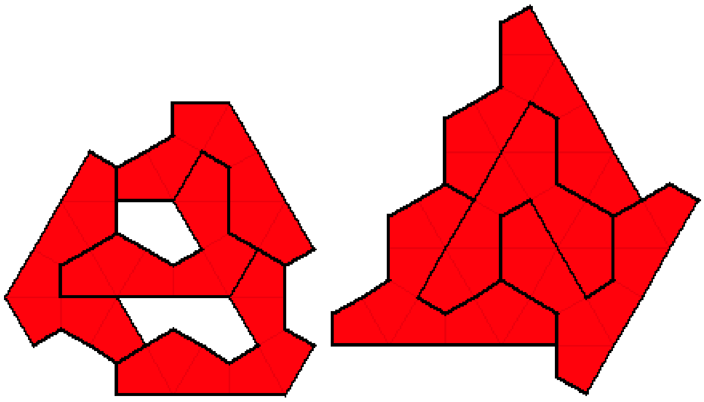


Fig. 7.

Besides, an error was found in the page „*Polyform Curiosities*" – the list of tetrads consisting of 10-*monds* was not complete. We informed G. Sicherman about the error and the lost tetrad was added.

3.2. I-trimino networks.

Problem 3. A problem was studied about the minimal I-trimino networks. The approach to the problem was the following: what minimal number of I-triminos can be inserted in the quadrangle $n \times n$, so that one more I-trimino could not be inserted in it. It is hard to algorithmize the problem, several approaches were tested and computer programs developed, but no considerable progress is achieved. Squares with the length of sides not exceeding 11 are analysed. The algorithm is exponential in time. For example, to find that it is not enough with a 23 I-trimino for a square of 11 × 11 the computerized proof takes about ~ 7 hours. The minimal network of such square consists of a 24 I-trimino. It is planned to research this problem still; it would be useful at least in two aspects: 1) pupils' scientific research projects are developed on the topic; 2) the sequence of the minimal I-triminos number is not still found in the popular encyclopaedia of sequences *The On-Line Encyclopaedia of Integer Sequences* [http://oeis.org].

Publications

[1.] **M.Avotiņa**. Inequalities in mathematical olympiads. In: Proceedings of The 6th International conference on creativity in mathematics education and the education of gifted students, University of Latvia, Latvia/ Angel Kanchev University of Ruse, Bulgaria, 2010. – pp. 7 – 12.
[2.] **D. Bonka, Z. Kaibe**. Mathematikwettbewerbe für die Schüler in Lettland. – In: Beiträge zum Mathematikunterricht 2011, Band I, WTM, Münster, 2011. – S. 123 – 126.

Conferences theses

[1.] **M.Avotina** *Inequalities in mathematical olympiads*. - 6th International conference on creativity in mathematics education and the education of gifted students, Riga, Latvia, August 1-5, 2010.
[2.] **D. Bonka, Z. Kaibe**. *Mathematikwettbewerbe für die Schüler in Lettland.* – 45. Jahrestagung für Didaktik der Mathamatik, Freiburg, Deutschland, 21. – 25. February, 2011.
[3.] **D. Bonka**. *How to work with mathematically gifted students.* - 12th International conference Teaching Mathematics: Retrospective and Perspectives, Šiauliai, May 5 – 6, 2011.
[4.] **M.Avotiņa**. *Difference Equations in School Curriculum and Mathematical Contests.* - 12th International conference Teaching Mathematics: Retrospective and Perspectives, Šiauliai, May 5 – 6, 2011.
[5.] **D. Bonka**. *On Problem Set's composition for math olympiads.* – 13th International conference Teaching Mathematics: Retrospective and Perspectives and 8th Nordic-Baltic conference AGROMETRICS, Tartu, 30 May-01 June, 2012.
[6.] **D. Bonka** *Does Science Help in Advanced Math Education?* – The 7th International conference on Creativity in Mathematics Education and the Education of Gifted Students, Korea Sciebce Academy of KAIST, July 15-18, 2012.
[7.] **J.Čerņenoks**. *Polyominoes as a Rich Source for an Appropriate Research Topics for Gifted Students*. - The 7th International conference on Creativity in Mathematics Education and the Education of Gifted Students, Korea Sciebce Academy of KAIST, July 15-18, 2012.
[8.] **D. Bonka**. Member of the Discussion group DG 2: Creativity in Mathematics Education at the International congress on Math Education ICME 2012, Seoul, South Korea, July 8-15, 2012.